Weighted averages of ℓ^p sequences

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Motivation

Weighted Dirichlet spaces

Definition

Let ω be a positive superharmonic function on $\mathbb{D}$ and $f \in \mathrm{Hol}(\mathbb{D})$. We define

$$\mathcal{D}_\omega(f) := \int_{\mathbb{D}} |f'(z)|^2 \omega(z) \mathrm{d}A(z),$$

where $\mathrm{d}A$ denotes normalized area measure on $\mathbb{D}$. The *weighted Dirichlet space* $\mathcal{D}_\omega$ is the set of functions $f \in \mathrm{Hol}(\mathbb{D})$ such that $\mathcal{D}_\omega(f) < \infty$.

Motivation

Hadamard multipliers

Definition

Let $f(z) := \sum_{k=0}^{\infty} a_k z^k$ and $g(z) := \sum_{k=0}^{\infty} b_k z^k$ be two formal power series. Their *Hadamard product* is defined to be

$$(f * g)(z) := \sum_{k=0}^{\infty} a_k b_k z^k.$$

The *Hadamard mulipliers* of $\mathcal{D}_\omega$ are the formal power series h that have the property that $h * f \in \mathcal{D}_\omega$ for each $f \in \mathcal{D}_\omega$.

Motivation

Definition of T_h

Definition

Let $h(z) := \sum_{k=0}^{\infty} c_k z^k$. We define

$$T_h := \begin{pmatrix} c_1 & c_2 - c_1 & c_3 - c_2 & c_4 - c_3 & \dots \\ 0 & c_2 & c_3 - c_2 & c_4 - c_3 & \dots \\ 0 & 0 & c_3 & c_4 - c_3 & \dots \\ 0 & 0 & 0 & c_4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Motivation

Characterization of the Hadamard multipliers of $\mathcal{D}_\omega$

Theorem (Mashreghi–Ransford, 2019)

Let $h(z)$ be a formal power series. The following statements are equivalent.
(i) h is an Hadamard multiplier of $\mathcal{D}_\omega$ for every superharmonic weight ω.
(ii) T_h acts as a bounded operator on ℓ^2.

Moreover, $\mathcal{D}_\omega(h * f) \leq \|T_h\|^2_{\ell^2 \to \ell^2} \mathcal{D}_\omega(f)$ for every superharmonic weight ω and $\|T_h\|^2_{\ell^2 \to \ell^2}$ is the best possible constant.

The L-matrices

Definition

Let (a_n) be a sequence of complex number. An L-matrix is an infinite matrix of the form

$$A := [a_n] = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_1 & a_2 & a_3 & \cdots \\ a_2 & a_2 & a_2 & a_3 & \cdots \\ a_3 & a_3 & a_3 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

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Question : When does the L-matrix $[a_n]$ act as a bounded operator on ℓ^2 ?

Partial answer

Corollary (B., Mashreghi ; 2021)

Let $A = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \dots \\ a_1 & a_1 & a_2 & a_3 & \dots \\ a_2 & a_2 & a_2 & a_3 & \dots \\ a_3 & a_3 & a_3 & a_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$ *be an infinite positive L-matrix. For A to act as a bounded operator on* ℓ^2*, the condition* $a_n = O\left(\frac{1}{n^\alpha}\right)$ *is*

- *necessary if* $\alpha \leq \frac{1}{2}$ *;*
- *neither necessary nor sufficient if* $\frac{1}{2} < \alpha < 1$ *;*
- *sufficient if* $\alpha \geq 1$*.*

A particular inequality

Our initial goal

$$\sum_{n=1}^{\infty} \frac{1}{4^n} \left| \sum_{k=1}^{4^n} a_k \right|^2 \leq C \sum_{n=1}^{\infty} |a_n|^2.$$

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$$\Updownarrow$$

Hardy's inequality (Special case)

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \left| \sum_{k=1}^{n} a_k \right|^2 \leq C \sum_{n=1}^{\infty} |a_n|^2.$$

Hilbert's inequality

Theorem (Hilbert ; 1906)

If (a_m) and (b_n) are sequences of positive numbers, then

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n} \leq 2\pi \left(\sum_{m=1}^{\infty} a_m^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} b_n^2 \right)^{\frac{1}{2}}.$$

David Hilbert

Hilbert's inequality (Generalization)

Theorem (G.H. Hardy, M. Riesz ; 1920)

If (a_m) and (b_n) are sequences of positive numbers and $p \in (1, \infty)$, then

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n} \leq \frac{\pi}{\sin(\frac{\pi}{p})} \left(\sum_{m=1}^{\infty} a_m^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{q}}.$$

Moreover, the constant $\pi \operatorname{cosec}(\frac{\pi}{p})$ is optimal.

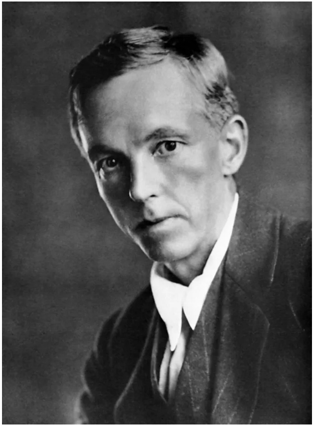
Marcel Riesz

Hardy's inequality

Theorem (Hardy, Riesz ; 1920)

If (a_n) *is a sequence of positive numbers and* $p \in (1, \infty)$*, then*

$$\sum_{n=1}^{\infty} \left(\frac{a_1 + a_2 + \cdots + a_n}{n} \right)^p \leq \left(\frac{p^2}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p.$$



G.H. Hardy

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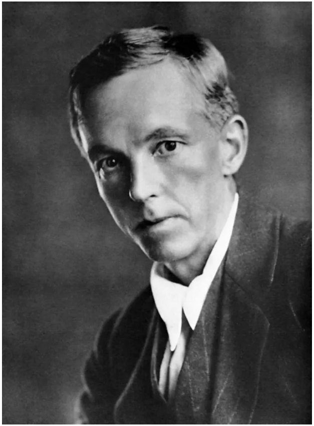
Question : Is the constant $\left(\frac{p^2}{p-1}\right)^p$ optimal ?



G.H. Hardy

Hardy's inequality (Optimal constant)

Theorem (E. Landau ; 1926)

If (a_n) *is a sequence of positive numbers and* $p \in (1, \infty)$, *then*

$$\sum_{n=1}^{\infty} \left(\frac{a_1 + a_2 + \cdots + a_n}{n} \right)^p \leq \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p.$$

Moreover, the constant $\left(\frac{p}{p-1}\right)^p$ *is optimal.*

Edmund Landau

My supervisors and I

Javad Mashreghi

Frédéric Morneau-Guérin

My cat and I

The main goal

Hardy's inequality

$$\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} 1 \cdot a_k \right|^p \leq C \sum_{n=1}^{\infty} a_n^p.$$

The main goal

Our inequality

$$\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k \in \mathcal{N}_n} 1 \cdot a_k \right|^p \leq C \sum_{n=1}^{\infty} a_n^p.$$

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$$\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k \in \mathcal{N}_n} m_k a_k \right|^p \leq C \sum_{n=1}^{\infty} a_n^p.$$

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$$\sum_{n=1}^{\infty} \left| \frac{1}{M_n} \sum_{k \in \mathcal{N}_n} m_k a_k \right|^p \leq C \sum_{n=1}^{\infty} a_n^p .$$

Definitions

Summation indices

- Let $\mathbb{N} = N_1 \cup N_2 \cup \cdots$ be a partition of $\mathbb{N}$.

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Example

Let $N_1 = \{1, 2, 3\}$ and $N_n = \{2^{n-1} + 1, 2^{n-1} + 2, \ldots, 2^n\}$ if $n \geq 2$. Then

$$\mathcal{N}_1 = \{1, 2, 3\}, \quad \mathcal{N}_n = \{1, 2, \ldots, 2^n\} \quad \text{if } n \geq 2.$$

Definitions

Weights

• $\mathbb{N} = N_1 \cup N_2 \cup \cdots$ • $\mathcal{N}_n := N_1 \cup N_2 \cup \cdots \cup N_n$

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- $\rho := \sup_{n\geq 1} \left(w_n \sum_{k=n}^{\infty} \frac{1}{M_k} \right)$.

Main result

- $\mathbb{N} = N_1 \cup N_2 \cup \cdots$
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Theorem (B., Mashreghi, Morneau-Guérin ; 2023)

Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers, and let $(m_n)_{n\geq 1}$ be a sequence of weights. Define M_n and ρ as above, and assume that ρ is finite. Then

$$\sum_{n=1}^{\infty} \left| \frac{1}{M_n} \sum_{k \in \mathcal{N}_n} m_k a_k \right|^p \leq \rho \sum_{n=1}^{\infty} |a_n|^p.$$

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Theorem (B., Mashreghi, Morneau-Guérin ; 2023)

Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers, and let $(m_n)_{n\geq 1}$ and $(M_n)_{n\geq 1}$ be two sequences of weights. If

$$\rho := \sup_{n\geq 1} \left(w_n \sum_{k=n}^{\infty} \frac{(w_1 + \cdots + w_k)^{p-1}}{M_k^p} \right)$$

is finite, then

$$\sum_{n=1}^{\infty} \left| \frac{1}{M_n} \sum_{k \in \mathcal{N}_n} m_k a_k \right|^p \leq \rho \sum_{n=1}^{\infty} |a_n|^p.$$

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Example

If $N_n = \{n\}$, $m_n = 1$ and $M_n = n^{1+\varepsilon}$ $(\varepsilon > 0)$ for all $n \geq 1$, then

- $\mathcal{N}_n = \{1, 2, \ldots, n\}$;
- $w_n = 1$;
- $\rho = \sup_{n \geq 1} \sum_{k=n}^{\infty} \frac{1}{k^{1+p\varepsilon}} \leq \zeta(1 + p\varepsilon) < \infty$.

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$$\implies \sum_{n=1}^{\infty} \left| \frac{a_1 + \cdots + a_n}{n^{1+\varepsilon}} \right|^p \leq \zeta(1 + p\varepsilon) \sum_{n=1}^{\infty} |a_n|^p.$$

Lacunary sequence

Definition

A sequence $(n_k)_{k\geq 1}$ of positive integers satisfy the Hadamard gap condition if there exist some $r > 1$ such that

$$\frac{n_{k+1}}{n_k} \geq r, \qquad (k \geq 1).$$

Whenever this is the case, $(n_k)_{k\geq 1}$ is called a *lacunary sequence of ratio* r.

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Example

If $r > 1$, $(r^k)_{k\geq 1}$ is a lacunary sequence of ratio r.

Corollaries

An application to lacunary sequences

Corollary (B., Mashreghi, Morneau-Guérin ; 2023)

Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers, and let $(n_k)_{k\geq 1}$ be a lacunary sequence of ratio r. If $p \in (1, \infty)$, then

$$\sum_{k=1}^{\infty} \left| \frac{1}{n_k^{1/q}} \sum_{j=1}^{n_k} a_j \right|^p \leq \left(\frac{r^{1/q}}{r^{1/q}-1} \right)^p \sum_{n=1}^{\infty} |a_n|^p.$$

Corollaries

An extreme case : Geometric sequences

Theorem (B., Mashreghi, Morneau-Guérin ; 2023)

Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers and let $b \geq 2$ be an integer. Then

$$\sum_{k=1}^{\infty} \frac{1}{b^k} \left| \sum_{j=1}^{b^k} a_j \right|^2 \leq \frac{\sqrt{b}+1}{\sqrt{b}-1} \sum_{n=1}^{\infty} |a_n|^2.$$

Moreover, the constant $\frac{\sqrt{b}+1}{\sqrt{b}-1}$ is optimal and the above inequality is strict, except if $(a_n)_{n\geq 1}$ is the null sequence.

Corollaries

An extreme case : Geometric sequences

Example

If $b = 4$, then

$$\sum_{n=1}^{\infty} \frac{1}{4^n} \left| \sum_{k=1}^{4^n} a_k \right|^2 \leq 3 \sum_{n=1}^{\infty} |a_n|^2.$$

Moreover, the constant 3 is optimal.

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