Solving Vincent Carret's Puzzle:
A Rebuttal of Carret's Fallacies and Errors

Jean-Marc Ginoux
PhD in Mathematics, PhD in History of Science
https://www.researchgate.net/profile/Jean-Marc-Ginoux
Centre de Physique Théorique
Université de Toulon, France
ginoux@univ-tln.fr

Franck Jovanovic
PhD in Economics
https://www.researchgate.net/profile/Franck-Jovanovic
School of Business Administration
TELUQ University, Montreal, Canada

Laboratoire d'Economie d'Orléans
Université d'Orléans, France
franck.jovanovic@teluq.ca

Abstract:
The present rejoinder aims at rebutting Carret's allegation of mistaken interpretation in our work. We expose mathematical errors in Carret's work, also present in his publications with Assous. Then, and most importantly, we show the unfounded economic consequences that follow from erroneous mathematical assumptions about Frisch's model. We demonstrate that Carret's statements are based on a misunderstanding of Frisch's econometric model and approach. Then, we show that Carret's results are not supported by the demonstration he claims to have made, and that he misrepresents the arguments of some authors, making them say things they never said.

In a recent article that gave birth to a mini-symposium, we propose a solution to a controversy about Frisch's work opened 30 years ago by Stefano Zambelli (Ginoux and Jovanovic 2022c). In 1992 and 2007, Zambelli published a numerical analysis of one of the models developed by Frisch in his seminal book chapter "Propagation problems and impulse problems in dynamic economics." In the conclusion of his work, supported by his numerical analysis, Zambelli claimed that Frisch's "rocking horse model is not rocking!" Since then, several authors have taken up this conclusion with varying degrees of caution (Velupillai 1998; Louçã 2001; Bjerkholt 2007; Bjerkholt and Dupont 2010; Dupont-Kieffer 2012; Kolsrud and Nymoen 2014; Boumans 2020; Assous and Carret 2022; Carret 2020, 2022b). For instance, referring to Zambelli (2007), Bjerkholt and Dupont (2010, 53, fn 25) explained that "ironically, Frisch erred in his presentation. The model, which has been studied more than any other business cycle model, did not generate cycles." In the same vein, Assous and Carret (2022, 69) claimed that "[i]n fact, for his original parameters, the fluctuations will not appear at all in the aggregated propagation mechanism […] Zambelli (1992; 2007) was the first to notice it."

In our article, we proved that Zambelli's numerical analysis hides a mathematical error, and consequently does not hold (Ginoux and Jovanovic 2022c). Zambelli's mathematical error in turn conceals an economic error: the fact that different types of cycles (such as Kitchin and Juglar cycles) would impact the activity with the same amplitude. There is no economic justification for such a hypothesis, as explained for instance by Schumpeter (1939) and again for instance by Dal Pont Legrand and Hagemann (2007, 12). Therefore, Zambelli's numerical analysis has no value for understanding economic theoretical debates that were outgoing at the time when Frisch published his book chapter. Zambelli declined the editor's invitation to participate in the mini-symposium and respond to our demonstration (Boianovsky 2022). Is this an admission that he understood that there was indeed an error in his work? In any case, this mini-symposium gave another specialist, Lionello Punzo (2022, 174), an opportunity to confirm that Zambelli did not prove his claim.

While we were working on this topic, Michael Assous and Vincent Carret (2021, 2022) as well as Carret (2020, 2022a, 2022b) published several articles and a book on the early works in macrodynamic. Like Zambelli, they used numerical analysis for studying the publications of the first econometricians, particularly Frisch's. They also introduced new mathematical approach for supporting some of their results, in particular the Laplace transform.

In their publications, Assous and Carret have created a new controversy in line with Zambelli's puzzle. They argued that they were the first authors to find an error in Frisch's model, which has been analyzed by some of the most important economists, including several recipients of the Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel. What is this new controversy about? Assous and Carret started from Zambelli's puzzle and claimed that it can be solved by modifying the Frisch parameters and by preventing Frisch's error. They support their claim with mathematical and numerical analyses. Unfortunately, in the light of our recently published demonstration, their attempt to modify the Frisch parameters to show that his model can oscillate seems pointless, since Frisch's model rocks with its original parameters.

In his comment on our criticism of Zambelli's work, in order to defend his own results against our demonstration, Carret argued that we had not correctly understood some mathematical elements of Frisch's work (Carret 2022a). In his article published by *JHET* in 2022, in footnote 19 page 9, Carret reiterates this statement and claims that our result "is based on a mistaken interpretation of the paragraph at the bottom of p. 191 of Frisch (1933)." He then states that we

> "take to mean that the coefficient of each cycle in the general sum of solutions is arbitrary, while […] these coefficients [depended] on initial conditions and the parameters of the system."

To support his claim, Carret uses mathematical explanations that may seem intimidating for non-specialists.

These mathematical questions, which at first glance are very technical, are an invitation to take the time to analyze the assumptions, concepts and economic reasoning behind mathematical developments or numerical simulations. Analyzing the work of early econometricians by going into the mathematical details provides an opportunity to ensure that the mathematical developments or numerical simulations used by previous and contemporary authors are not mere tricks but have a verified economic foundation.

Doing such an analysis, led us to deduce results in history of economics for the early econometricians. We have also done the same for Assous and Carret's work, since Carret claims that we do not understand the mathematics we use and that do not understand Frisch's model.

The present rejoinder aims at rebutting Carret's allegation of mistaken interpretation in our work. We expose mathematical errors in Carret's work, also present in his publications with Assous. Then, and most importantly, we show the unfounded economic consequences that follow from erroneous mathematical assumptions about Frisch's model. We demonstrate that Carret's statements are based on a misunderstanding of Frisch's econometric model and approach. Then, we show that Carret's results are not supported by the demonstration he claims to have made, and that he misrepresents the arguments of some authors, making them say things they never said.

A shorter version of this rejoinder will be published by *JHET*, but due to space limitation, we have decided here to give access to this longer version.

1. Carret ignored Frisch's closure relation

In his criticism of our work, Carret states that we made a mistaken interpretation of the following paragraph:

> "A given set (18) (for a given *j*) does not –taken by itself– satisfy the dynamic system consisting of (3.3), (2) and (4). It will do so only if the structural constant $c = 0$. If $c \neq 0$ the constant terms a_*, b_* and c_* must be added to (18) in order to get a correct solution. If these constant terms are added, we get functions that satisfy the dynamic system, and that have the property that any linear combination of them (with constant coefficients) satisfy the dynamic system provided only that the sum of the coefficients by which they are linearly combined is equal to unity. This proviso is necessary because any sets of functions that shall satisfy the dynamic system must have the uniquely determined constants a_*, b_* and c_*." (Frisch 1933, 191)

In this paragraph, Frisch expressed a condition that must be respected in order to obtain the general solution (i.e., when $c \neq 0$). He called it a "proviso." According to the *Cambridge Dictionary*, a proviso is "a statement in an agreement, saying that **a particular thing must happen before another can**."

Carret has never explained why he has simply ignored Frisch's proviso in his publications. So we can suspect that Carret has not understood the economic reasons for which Frisch applied it. If Frisch wrote a proviso, it is not to be ignored. Ignoring a condition that must be taken into account is a significant error in reasoning. For instance, if we write that, for a real number x, $2 + 2 + x = 5$, proviso $x = 1$. Ignoring this proviso generates a mathematical error, which creates economic errors; it has nothing to do with a problem of interpretation as some might claim.

What does Frisch's proviso mean? In modern terms, it is a closure relation, according to which the sum of the coefficients k_j, which are the weight of each cycle, must be equal to unity. With this closure relation, Frisch normalized the weight of each cycle. The general solution of his propagation model reads:

$$
\begin{aligned}
x(t) &= x_0 + k_1 x_1 + k_2 x_2 + k_3 x_3 + \cdots \\
y(t) &= y_0 + k_1 y_1 + k_2 y_2 + k_3 y_3 + \cdots \\
z(t) &= z_0 + k_1 z_1 + k_2 z_2 + k_3 z_3 + \cdots
\end{aligned}
\tag{1}
$$

with $\sum_{j=1}^{\infty} k_j = 1$.

In ignoring Frisch's closure relation, Carret and Zambelli used $k_1 = k_2 = k_3 = \ldots = 1$. They did not mention this clearly anywhere, but we were able to establish it from their figures. In so doing, they fail to weigh the impact of different economic cycles. Does Carret's hypothesis make sense from an economic viewpoint? Does it make sense to state that different cycles (such as Kitchin and Juglar cycles) impact the activity with the same amplitude? Since these types of cycles do not have the same economic origins (Schumpeter 1939), there is no economic justification for such a hypothesis.

We are not alone in saying that this closure relation must be considered in Frisch's work. Lionello Punzo (2022) says the same thing. By respecting the closure relation, we proved that Frisch's model oscillates with its original parameters (Ginoux and Jovanovic 2022b, 2022c). Indeed, while Frisch did not provide the value of the coefficients k_j (because he did not give an explicit general solution of his model), we provide a general solution by choosing a set of coefficients that respect Frisch's closure relation and for which his propagation model fluctuates.[1]

Since Carret has not understood the role of this closure relation in Frisch's demonstration, he is led to assert that we would have taken the values of our coefficients arbitrarily without considering Frisch's initial conditions. This statement is erroneous. To show this, let us prove that Frisch's closure relation depends on his initial conditions.

[1] Our article (Ginoux and Jovanovic 2022a) contains a typographical error corrected by an erratum.

Let us write a linear combination of Frisch's *cyclical components* [eq. 18, Frisch, 1933, p. 190]. For the sake of simplicity, we will consider only the first variable, although the proof can be extended to all the others. We have thus:

$$x^C(t) = \sum_{j=1}^{\infty} k_j x_j(t) = \sum_{j=1}^{\infty} k_j A_j e^{-\beta_j t} \sin\left(\phi_j + \alpha_j t\right) \tag{2}$$

where k_j are the coefficients involved in the linear combinations. So, we have to prove that:

$$\boxed{\sum_{j=1}^{\infty} k_j = 1} \tag{3}$$

Using one of Frisch's *initial conditions*, i.e., $\dot{x}^C(0) = \frac{1}{2}$ (Frisch, 1933, p. 190), we find:

$$\dot{x}^C(0) = \sum_{j=1}^{\infty} k_j A_j \left(-\beta_j\right) e^{-\beta_j \times 0} \sin\left(\phi_j + \alpha_j \times 0\right) + \sum_{j=1}^{\infty} k_j A_j \alpha_j e^{-\beta_j \times 0} \cos\left(\phi_j + \alpha_j \times 0\right) = \frac{1}{2}$$

Frisch (1933, p. 190) also uses the *initial condition* $x^C(0) = 0$ which implies that: $\phi_j = 0$.

So, we have:

$$\dot{x}^C(0) = \sum_{j=1}^{\infty} k_j A_j \left(-\beta_j\right) e^{-\beta_j \times 0} \sin\left(0 + \alpha_j \times 0\right) + \sum_{j=1}^{\infty} k_j A_j \alpha_j e^{-\beta_j \times 0} \cos\left(0 + \alpha_j \times 0\right) = \frac{1}{2}$$

Then, we obtain:

$$\dot{x}^C(0) = \sum_{j=1}^{\infty} k_j A_j \left(-\beta_j\right) \underbrace{\sin(0)}_{=0} + \sum_{j=1}^{\infty} k_j A_j \alpha_j \underbrace{\cos(0)}_{=1} = \frac{1}{2}$$

It leads to $\sum_{j=1}^{\infty} k_j A_j \alpha_j = \frac{1}{2}$. But according to Frisch (1933, p. 190), we have:

$$A_j = \frac{1}{2\alpha_j} \quad \Leftrightarrow \quad A_j \alpha_j = \frac{1}{2}$$

It follows immediately that $\sum_{j=1}^{\infty} k_j = 1$.

This demonstration proves that Frisch's closure relation depends on his initial conditions, which Carret fails to understand. Thus, since the value of our coefficients respect Frisch's closure relation, they necessarily respect Frisch's initial conditions and cannot be arbitrary. Consequently, so far, Carret has failed to demonstrate that we have misunderstood Frisch's proviso. Moreover, in ignoring the role of a closure condition (or closure relation), Carret's results are invalid.

2. Carret ignored a well-known theorem

In footnote 19, Carret refers to another article (2022a) in which he claimed to have demonstrated our mistake. Carret (2022a, 168) states that Frisch's paragraph page 191 refers to the resolution of a system with a homogenous part and a non-homogeneous part. He clarifies by stating that

> "[a] combination of a homogeneous solution [i.e., Frisch's eq. 18] and a particular solution [i.e., a_*, b_* and c_*] will thus solve the system,"

and

> "the functions in [Frisch's eq.] (16) solved [the nonhomogeneous system] because they were composed of the sum of the particular solutions a_*, b_* and c_* and the trends which solved respectively the nonhomogeneous and the homogeneous parts of the system."

Then, to explain Frisch's proviso, Carret claims that

> "[w]hat Frisch was saying was that, when adding two of the solutions […], for instance x_0 [$x_0 = a_* + a_0 e^{\rho_0 t}$] and g_1 [$g_1 = a_* + A_1 e^{-\beta_1 t}\sin\left(\phi_1 + \alpha_1 t\right)$], we should be careful to end up with only one particular solution a_*.
>
> Suppose we do this addition; using arbitrary coefficients r_0 and r_1, we obtain
>
> $$r_0 x_0 + r_1 x_1 = a_*\left(r_0 + r_1\right) + r_0 a_0 e^{\rho_0 t} + r_1 A_1 e^{-\beta_1 t}\sin\left(\phi_1 + \alpha_1 t\right).$$
>
> It clearly appears that the coefficients r_0 and r_1 must sum to 1 so that we obtain only one particular solution a_* solving the nonhomogeneous part of the system."

Obviously, Carret is referring indirectly to the so-called the *Superposition Theorem*.

Unfortunately, Carret has misunderstood this theorem and applied it poorly. Indeed, this theorem is "an *existence and a uniqueness theorem*" (Tenenbaum and Pollard 1985, 208). It states that we must look:
- first, for a *particular solution* to the non-homogeneous equation for $c \neq 0$;
-second, for a *general solution* to the homogeneous equation for $c = 0$, which is given by a linear combination of linearly independent solutions.

- and third, for the general solution of the non-homogeneous equation which will be equal to **the sum of the *particular* and *general* solutions** (Tenenbaum and Pollard 1985, 208; Warusfel 1966, 138).

Given that a_* is a *particular solution*, and $a_0 e^{\rho_0 t}$ and $A_1 e^{-\beta_1 t}\sin(\phi_1+\alpha_1 t)$ are *general solutions*, Carret clearly did not respect the theorem. Indeed, he made a **linear combination of a mixture of both *particular* and *general* solutions** that led him to count twice the *particular solution*, which we never did. The *Superposition Theorem* requires precisely that the *particular solution* be counted only once.

In fact, Carret should have written the *general solution* of Frisch's system as:[2]

$$x(t) = \underbrace{a_*}_{(P)} + \underbrace{x_0^H(t) + x_C^H(t)}_{(H)} = \underbrace{a_* + a_0 e^{-\beta_0 t}}_{(T)} + \underbrace{\sum_{j=1}^{\infty} k_j A_j e^{-\beta_j t}\sin\left(\phi_j + \alpha_j t\right)}_{(C)} \tag{4}$$

where a_* is the *particular solution* while $x_0^H(t) = a_0 e^{\rho_0 t} = a_0 e^{-\beta_0 t}$ and $x_C^H(t) = \sum_{j=1}^{\infty} k_j A_j e^{-\beta_j t}\sin\left(\phi_j + \alpha_j t\right)$ are the *general solutions* of the homogeneous equation. This latter is a linear combination of Frisch's *cyclical components* (as demonstrated online).

By claiming that we have misinterpreted Frisch's paragraph, Carret attempts to draw readers into a false debate. Careful readers will have noted that in his criticism, Carret (2022a, 168) did not discuss the fact that "the sum of the coefficients [must be] equal to unity" in order to add the constant terms a_*, b_* and c_* "to (18) in order to get a correct solution." We suspect that Carret did not want to discuss the coefficients k_j, because this would lead him to question the *superposition theorem* and consequently the existence of a general solution in Frisch's model. Especially since the superposition theorem applies even when using the Laplace transform. Therefore, it is no coincidence that Carret criticizes our work on this paragraph: as soon as we respect Frisch's closure relation, which follows from the *superposition theorem*, Carret's whole demonstration collapses and his "results" on Frisch's model would have to be abandoned.

3. Solving Carret's puzzle

Now let us solve Carret's puzzle about the error he purports to have demonstrated in Frisch's work. As Carret has rightly mentioned in all of his publications, Frisch gave different initial conditions for the trends and the cycles. Carret claims that this was a serious error:

> "Frisch erred, because he gave different initial conditions for the trend and for the cycles, even though they should depend on the same initial development" (Carret 2022a, 165).

[2] Note that Frisch (1933, 188, eq. 16) could write the trend like this because the characteristic exponent of $a_0 e^{\rho_0 t}$ has no imaginary part.

This problem is crucial for Carret, since he has built his argumentation on this element and presents it as one of his major contributions:

> "[w]ith the hindsight of a more complete theory of mixed differential–difference equations, we can show analytically by using the Laplace transform that all components, whether cyclical or oscillatory, will depend on the same initial conditions" (Assous and Carret 2022, 67).

In fact, there is no error in Frisch's work. In the general case, Carret is correct: we should have the same initial condition for the trend and the cyclical components. But Frisch managed to find a particular case that works. Punzo (2022, 174) also pointed this out, leading him to claim that it is "almost an honor" for Frisch to "find one such constellation", *i.e.*, the three cyclical components and their periods. Carret has missed this in all of his publications, overlooking the originality of Frisch's work. Indeed, this "honor" results from Frisch's calibration with his econometric model.

This result demonstrates that Carret's work has no merit; he attempts to make the reader believe that he has shown something that is not in Frisch's writings, but this is not true.

4. Carret's work is useless

Carret claims that

> "unlike what Zambelli affirmed, it is possible to obtain the kind of fluctuations that Frisch described in his article after a slight change of parameters. This is important, because it shows that the conception of fluctuations and propagation advocated by Frisch was possible" (2022b, 9).

Unfortunately, as demonstrated in Ginoux and Jovanovic (2022c), Zambelli's assertion is based on a mathematical error. Punzo (2022) validated our result by claiming that Zambelli failed to prove that "the rocking horse does not rock." Consequently, Zambelli's assertion is pure speculation, without any mathematical or numerical demonstration. Taking Zambelli's work as a starting point, Carret reproduced the same errors. Moreover, as clarified by Ginoux and Jovanovic (2022b), it was because Carret (2022b) ignored Frisch's closure relation that he was obliged to change the value of the parameters to obtain oscillations. By doing so, in all his publications, Carret has simply ignored the economic problem that Frisch faced.

Indeed, the challenge that Frisch had to take up with his model was to reproduce two cycles observed at that time in the literature, i.e., the primary cycle of 8.57 years and the secondary cycle of 3.50 years (the tertiary cycle of 2.20 years is a prediction made by Frisch). So, Frisch wanted to calibrate an econometric model, which oscillates by construction (because of the *damped sine curves*), in order to reproduce the two cycles observed in the literature. Carret addressed a different problem. Instead of calibrating an econometric model to explain the observed cycles, as Frisch and we did, Carret demonstrated that a model more or less close to Frisch's (but not Frisch's) oscillates, provided that the values of Frisch's parameters are changed. Consequently, Carret's econometric model does not allow the reproduction of the frequencies of these two observed cycles (Table 1).

Table 1. Comparison between the results obtained

	Frequency				Damping factor			
	j=0 (trend)	j =1 (cycle 1)	j =2 (cycle 2)	j =3 (cycle 3)	j =0 (trend)	j =1 (cycle 1)	j =2 (cycle 2)	j =3 (cycle 3)
Carret (2022, 11)	--	6.5	3.2	2.1	0.084982*	0.043802*	0.08884*	0.13493*
Frisch (1933, 187, table 1)	--	8.5654	3.4950	2.2021	-0.08045	0.371335	0.5157	0.59105
Ginoux and Jovanovic	--	8.5654	3.4950	2.2021	-0.08045	0.371335	0.5157	0.59105

*: these data are neither computed nor provided in Assous and Carret's publications; we have computed them starting from Frisch's characteristic equations (1933, 184, eqs. 10, 12 and 13).

Carret (2022b, 11) obtained "a primary cycle with a period of about 6.5 years, a secondary cycle with a period of about 3.2 years … all values rather close to those in Frisch's article." In Carret's view, such differences do not represent an issue and he

> "do[es] not think that it necessarily is [a problem]" (2022b, 12).

Let us be serious: an 8.5-year cycle is very different from a 6.5-year cycle: over 20 years, we will have two cycles, and therefore two economic recessions, in one case and three in the other. That is very different.

Carret (2022b, 13) admits that

> "[t]here is, however, one caveat, compared with Frisch's original article: in order to obtain apparent cycles at the aggregate level, we had to decrease the damping of the system. In fact, the return to equilibrium is much longer than in Frisch's original article."

As we can see in Table 2, all Carret's parameters are different from Frisch's, except ε.

Table 2. Comparison between the parameters used

Carret (2022)	λ = 0.3	r = 1	s = 2	m = 1	μ = 15	ε = 6	c = 0,165**
Frisch (1933)	λ = 0.05	r = 2	s = 1	m = 0.5	μ = 10	ε = 6	c = 0,165
Ginoux and Jovanovic (2022)	λ = 0.05	r = 2	s = 1	m = 0.5	μ = 10	ε = 6	c = 0,165

**: Carret did not provide the value he used for c in his demonstrations. However, we were able to calculate it based on the value he used for a_*. According to his fig. 1, page 11, $a_* = 0{,}1833$. Then, since Frisch (1933, 188, eq. 17) stated $c = \lambda a_*(r + sm)$, we have $c = 0{,}3 * 0{,}1833 * (1 + 2 * 1) = 0{,}165$.

To make his general solution oscillate for a while, Carret tuned the values of Frisch's parameters, and left out elements of Frisch's demonstration when it did not suit him. For instance, his parameter λ is 6 times higher than Frisch's in order to increase the importance of the *transient regime* until reaching the stationary level (horizontal asymptote at 0.18 in Carret's Figure 1 & 2) which obviously does not oscillate around it. To show the issue with Carret's choices, we solved Frisch's characteristic equations (Frisch, Eqs. 10, 12 & 13, p. 184) using Carret's parameters. Using *Mathematica,* we calculated Carret's "damping exponent" to compare it with Frisch's (the files are available on request to the authors). We obtained for the first cyclical component (j = 1) the "frequency" $\alpha_1 = 0.95987$ and the "damping exponent" $\beta_1 = 0.0438$. By calculation we verified Carret's result for the period of the first cycle: $p_1 = 2\pi/\alpha_1 = 6.54$ years. With his own parameters Frisch (1933, 187, Table 1) obtained for the first cyclical component ($j = 1$), the "frequency" $\alpha_1 = 0.73355$ and the "damping exponent" $\beta_1 = 0.371335$. So, Frisch obtained for the first cycle period: $p_1 = 2\pi/\alpha_1 = 8.5654$ years. As we can see, Carret's "damping exponent" for the first cycle is 8.5 times lower than Frisch's (see Table 1). It means that Carret's general solution takes 8.5 times more time to damp, as seen in Carret's figures 1 & 2 (see the horizontal axis which extends for a century!) What economic significance should be made of an econometric model that takes 100 years to return to the stationary level? Did we observe such behavior in Frisch's time? Not at all.

Carret's work is useless for understanding Frisch's because he worked on a solution of Frisch's model that is different from the original one and which does not make it possible to reproduce the observed cycles that Frisch sought to reproduce. Moreover, because his model does not have the same economic behavior as Frisch's, his results cannot be directly compared with those of Frisch.

5. Carret's demonstration is incomplete and unverifiable

Carret's criticism hides a major methodological difference between our work and his: Carret does not give the information needed to reproduce his work; we do. Therefore, anyone can verify our conclusions, but no one can verify Carret's work. For proof, in footnote 19 of his *JHET* article, Carret criticizes our work by invoking his "solution based on the Laplace transform" and its inverse. However, he does not provide the demonstration of his analytical solution. He explains that

> "[the] full derivation of this solution is published in Assous and Carret (2022)."

Unfortunately, Assous and Carret (2022) did not provide the calculations of the inverse Laplace transform they used. Carret's work is therefore unverifiable. This is embarrassing because the Laplace transform and its inverse underpin all Carret's analysis and arguments. It is on this basis that he could claim that Frisch "erred," or made "an error," or that he "can give a more elegant answer than Frisch" (Carret 2022b, 9). Moreover, according to Carret, the Laplace transform and its inverse are "modern mathematical tools that [Frisch] did not know" (2022b, 3). This is an astonishing claim to make, given that the Laplace transform was introduced in 1737, that the first use of its modern formulation dates back to 1910, and that in "the 1920s and 1930s it was seen as a topic of front-line research" (Deakin 1992, 265).

This is not our only issue with Carret's "solution based on the Laplace transform." As Allen (1959, 155-6) explained, the Laplace transform is a "trick" of mathematicians. One of the main problems with this trick is that when we use the Laplace transform and its inverse, we automatically introduce new constants (i.e., new initial conditions). Thus, "when the solution is obtained, it has the initial conditions ‘built in’”, and “*n* arbitrary constants, to be ‘fitted’ or evaluated with great labor from the initial conditions” (Allen 1959, 159). In other words, to solve a system similar to Frisch’s with a Laplace transform and its inverse, we have to generate at least one or two new initial conditions which are arbitrary by construction!

It is surprising, even shocking, that Carret keeps silent about this problem in his work, while he constantly criticizes Frisch on the value of his initial conditions. More vexatious is the fact that in none of his publications does Carret provide the value for his initial condition(s), including the new ones he introduced with the Laplace transform. It is legitimate to ask why he does not bother to provide the initial conditions he used in his work. By not providing his initial condition(s) and his inverse Laplace transform, Carret ensures that no one can reproduce and thus verify his work, as is expected in a scientific work.

6. Carret makes authors say what they do not say

Carret pretends that

> “[we] have suggested another approach [than his] to exhibit fluctuations in Frisch’s propagation mechanism” (2022b, 9 , fn. 19).

This statement is fallacious. Unlike Carret, we worked within Frisch’s framework (see Table 1), and strictly followed Frisch’s demonstration step by step without introducing any new mathematical tools or economic reasoning, as Carret does. In so doing, we proved that Frisch’s model fluctuates with its original values. This is not the first time that, to support his demonstrations, Carret has made authors say things they did not say or has pretended that authors did things they did not do. Here are a few other striking examples.

Assous and Carret (2022, 39, Fig. 3.1) reproduced a figure of Ludwig Hamburger (1930, 6), which he named ”Figure 2a,” and they added under the figure the title “Relaxation oscillations when $\alpha = 1$. *Source* Hamburger (1930: 6)” (see Fig. 1 below).

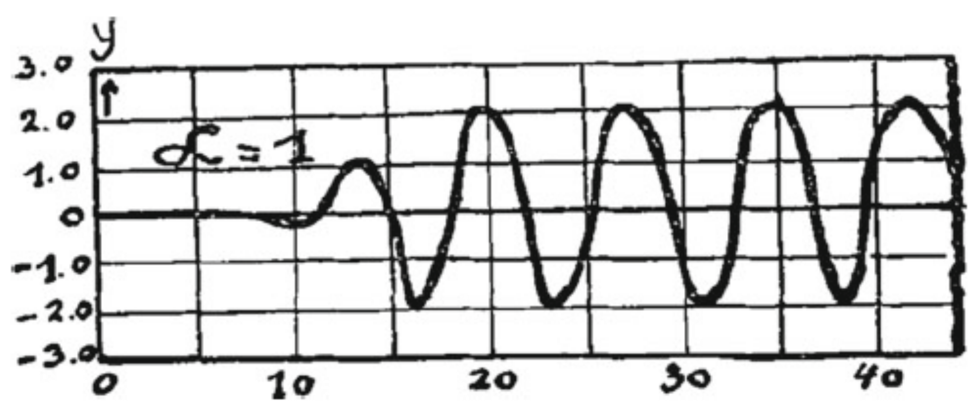

Fig. 3.1 Relaxation oscillation when $\alpha = 1$. *Source* Hamburger (1930: 6)

Fig. 1. Relaxation oscillations according to Assous and Carret (2022, 39, Fig. 3.1).

However, Hamburger did not specify the type of oscillations under his figure (see Fig. 2 below).

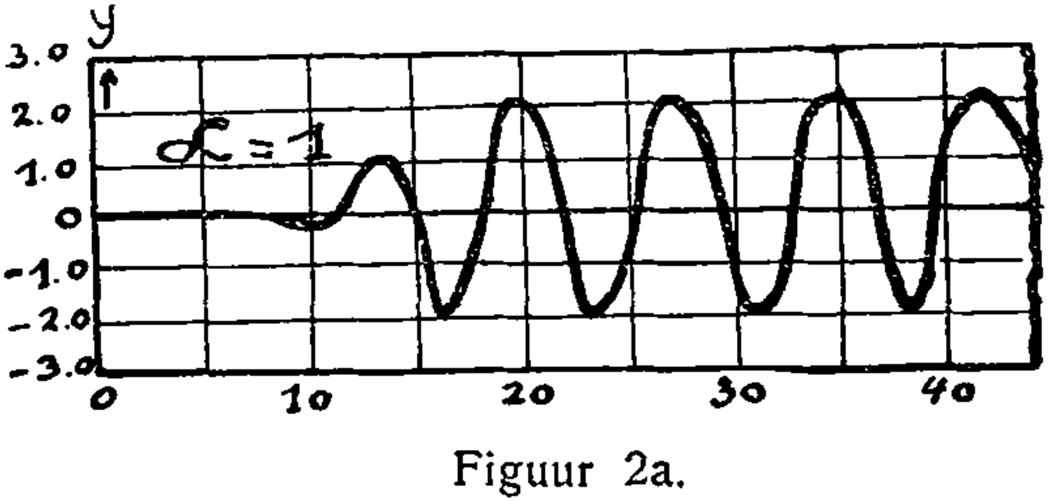

Fig. 2. Original figure from Hamburger (1930, 6).

A knowledgeable reader will see that it does not show relaxation oscillations; they are close to harmonic oscillations, as Hamburger clearly explained in his article. By attributing this identification to Hamburger, who never claimed that this figure represented relaxation oscillations, Assous and Carret outrageously mislead readers.

Second example: Assous and Carret (2022, 50-1) stated that

> "Goodwin … proposed to follow relaxation oscillations […]. Goodwin who was eventually the first to see two decades later how relaxation oscillation equations could be derived from the multiplier and accelerator mechanisms made no mention of Hamburger's works."

Unfortunately, contrary to Hamburger, Goodwin's model does not involve relaxation oscillations; Goodwin developed a model with self-maintained oscillations. As Goodwin (1951, 13) explained

> "the problem of the maintenance of oscillation was originally conceived by Lord Rayleigh and that our equation is of the Rayleigh, rather than the van der Pol."

Knowing that Van der Pol and Hamburger worked on relaxation oscillations, by linking Goodwin's model to Hamburger's work, Assous and Carret again mislead readers.

We could continue the list of misrepresentations and fallacious claims…

When Carret affirms that we (or anyone else) did something, readers should look closer. Moreover, since such erroneous affirmations help him to support his own demonstration, they are problematic in terms of scientific integrity.

Conclusion

By putting forward economic arguments that have the appearance of being based on mathematical analysis, Carret claims several things about our work, as well as Frisch's, that are simply false. Moreover, his criticisms are based on fallacious arguments that will mislead economists who are not familiar enough with mathematics. By ignoring Frisch's closure relation, Carret replicated Zambelli's error and continues to spread the same baseless arguments. By changing Frisch's parameters in his publications, Carret introduced additional new puzzles and additional economic errors, ignoring the relevance of the econometric approach defended by Frisch. Carret asserted things that are false and did not give all the information needed to verify his results.

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Appendix: General solution of Frisch's model

This appendix demonstrates the general solution of Frisch's model based on the superposition theorem.

In his book chapter, Frisch defined his mixed system of differential and difference equations as follows (the number in square brackets refers to the equation's number and the page in Frisch's book chapter):

$$\begin{cases} \dot{x} = c - \lambda\left(rx + sz\right) \\ y = mx + \mu\dot{x} \\ \varepsilon\dot{z} = y\left(t\right) - y\left(t - \varepsilon\right) \end{cases} \qquad \text{(A1) [eqs. 2, 3.3, 4, p. 177 \& 182]}$$

Due to the presence of the structural constant c, the first linear differential equation[3] of Frisch's system (A1) is non-homogeneous. Thus, if $c \neq 0$ this equation which can be written as follows:

$$\dot{x} + \lambda\left(rx + sz\right) = c \qquad \text{(A2)}$$

has for right-hand-side c and is called a non-homogeneous equation. If $c = 0$, eq. (A2) has for right-hand-side 0 and reads:

$$\dot{x} + \lambda\left(rx + sz\right) = 0 \qquad \text{(A3)}$$

Equation (A3) is then called a homogeneous equation.

Solving equation (A2), in the general case $c \neq 0$, requires the use of the so-called *Superposition Theorem* (Tenenbaum and Pollard 1985, 208, Theorem 19.3; Warusfel 1966, 138, [10]). This theorem states that we must look:
- first, for a *particular solution* to the non-homogeneous equation for $c \neq 0$;
-secondly, for a *general solution* to the homogeneous equation for $c = 0$, which is given by a linear combination of linearly independent solutions.
- and thirdly, for the general solution of the non-homogeneous equation which will be equal to the sum of the *particular* and *general* solutions.

So, let's compute the particular solution of the non-homogenous equation (A2) for $c \neq 0$.

1. **Particular solution of the non-homogenous ODE**

[3] In the following for sake of simplicity we will only write *equation* instead of *linear differential equation*.

The particular solution of (A2) for $c \neq 0$ is obtained by considering that both functions $x(t)$ and $z(t)$ are constant. This solution represents the so-called *stationary state* for which $\dot{x}(t) = 0$ and $\dot{z}(t) = 0$. This implies that $x(t) = a_*$ and $z(t) = c_*$, as recalled by Frisch (1933, 188):

> "Indeed, if $t \to \infty$ the functions (16) will approach the stationary levels a_*, b_* and c_*."

Thus, equation (A2) reads:

$$\lambda\left(ra_* + sc_*\right) = c \tag{A4}$$

This equation (A4) corresponds exactly to Frisch's third equation (1933, 188, eq. 17) and is thus verified. So, the *particular solution* of the non-homogeneous Frisch's system (1933, 177 & 182, eq. 2, 3.3, 4) reads:

$$\left(x^P(t), y^P(t), z^P(t)\right) = \left(a_*, b_*, c_*\right) \tag{A5}$$

Now, let's compute the *general solution* of the homogenous equation (A3) for $c = 0$.

2. General solution of the homogenous ODE

The general solution of (A3) for $c = 0$ is obtained by considering the functions $x(t)$ and $z(t)$ as "time series of the form"

$$\begin{cases} x^H(t) = \sum_{j=0}^{\infty} a_j e^{\rho_j t} \\ z^H(t) = \sum_{j=0}^{\infty} c_j e^{\rho_j t} \end{cases} \tag{A6}$$

where $\rho_j = -\beta_j + i\alpha_j$ with $i = \sqrt{-1}$ and $j = 0,1,2\ldots$ By solving the characteristic equation (1933, 184, eq. 12 & 13), Frisch deduced the four values of $\left(\beta_j, \alpha_j\right)$ for $j = 0,1,2,3$.

a. The case *j* = 0

For $j = 0$, he found that $\left(\beta_0, \alpha_0\right) = \left(0.08045, 0\right)$ and wrote the general solution of the homogeneous equation (A3) for $c = 0$ as:

$$x_0^H(t) = a_0 e^{\rho_0 t} = a_0 e^{-\beta_0 t} \tag{A7}$$

Let's notice that this general solution (A7) of the homogeneous equation (A3) for $c=0$ is the second part of what Frisch (1933, eqs. (18), p. 188) defined as the *trend* $x_0(t)=a_* + a_0 e^{\rho_0 t}$.

b. The cases *j* = 1, 2, 3.

For $j=1,2,3$, he found several values for β_j and α_j presented in his Tab. 1, Frisch (1933, 187) which were all different from zero and were corresponding to the *primary*, *secondary* and *tertiary cycle*. Frisch (1933, 190, eq. 18) wrote these *cyclical components* as follows:

$$\begin{cases} x_j^H(t)=A_j e^{-\beta_j t}\sin(\phi_j+\alpha_j t) \\ z_j^H(t)=C_j e^{-\beta_j t}\sin(\theta_j+\alpha_j t) \end{cases} \quad \text{(A8) [eq. 18, p. 190]}$$

Frisch (1933, 191) confirmed, as follows, that these equations (A8) are linearly independent solutions of the homogeneous (A3) for $c=0$:

"A given set (18) (for a given *j*) does not – taken by itself – satisfy the dynamic system consisting of (3.3), (2) and (4). It will do so only if the structural constant $c=0$."

But, according to the *Superposition Theorem* the general solution of a homogeneous equation (A3) for $c\neq 0$ is given by a linear combination of linearly independent solutions (A8).

So, the *general solution* of the homogeneous (A3) for $c=0$ is given by a linear combination of all the $x_j^H(t)$ (resp. $z_j^H(t)$) and can be written as:

$$x_C^H(t)=\sum_{j=1}^{\infty}k_j x_j^H(t)=\sum_{j=1}^{\infty}k_j A_j e^{-\beta_j t}\sin(\phi_j+\alpha_j t) \quad \text{(A9)}$$

where k_j are constant coefficients and subscript C means *cyclical components*. This result is also confirmed by Frisch (1933, 191):

"…we get functions that satisfy the dynamic system, and that have the property that any linear combination of them (with constant coefficients) satisfy the dynamic system…"

Then, the *general solution* of the homogeneous (A3) for $c=0$ and for $j=0,1,2,3$ reads:

$$x_0^H(t)+x_C^H(t)=a_0 e^{-\beta_0 t}+\sum_{j=1}^{\infty}k_j A_j e^{-\beta_j t}\sin(\phi_j+\alpha_j t) \quad \text{(A10)}$$

3. General solution of the non-homogenous (A2) for *c* ≠ 0

Although, Frisch never wrote the general solution of his system, let's give its expression for the first variable. According to the *Superposition Theorem*, the general solution of the non-homogeneous equation (A2) is equal to the sum of the particular solution (A5) and the general solution (A10). This gives for the *general solution* of Frisch's system:

$$x(t) = \underbrace{a_*}_{(P)} + \underbrace{x_0^H(t) + x_C^H(t)}_{(H)} = \underbrace{a_* + a_0 e^{-\beta_0 t}}_{(T)} + \underbrace{\sum_{j=1}^{\infty} k_j A_j e^{-\beta_j t} \sin\left(\phi_j + \alpha_j t\right)}_{(C)} \tag{A11}$$

This Appendix proves that Carret has not understood the *Superposition Theorem*, although he implicitly referred to it. Remember that, to solve Frisch's model, even with a Laplace transform, as Carret did, we need this theorem. A correct and rigorous application of the *Superposition Theorem* shows that Carret's demonstrations are false and baseless.