Benchmark, Relative Return, and Asset Pricing: An Extension

Claude Bergeron

School of Business Administration, Teluq University, Quebec, Canada

Claude Bergeron, École des sciences de l'administration, Université TÉLUQ,
455, rue du Parvis, Québec (Québec), Canada, G1K 9H6. E-mail: claude.bergeron@teluq.ca

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Abstract

In this working paper, we extend the benchmark asset pricing model of Bergeron (2021). We develop our extension model in two distinct situations. In the first case, we assume that the asset expected relative returns are identical. In the second case, we relax this restrictive assumption and suppose that these expected values could be unequal. This suggests that a benchmark asset pricing model can be built in different contexts.

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I. Introduction

The *benchmark approach* for the valuation of assets represents a growing field of research in *mathematical finance*. This approach was developed in a continuous-time framework by Platen (2006), and Platen and Heath (2006). Adopting this procedure, the capital asset pricing model (CAPM) of Sharpe (1964) and Lintner (1965) follows without expected utility functions, equilibrium conditions, or arbitrage restrictions (see also Karatzas and Kardaras, 2007; Platen and Rendek, 2012; Du and Platen, 2016; and Curchiero et al., 2019).

A long the same lines, Bergeron (2021) derived a simple asset pricing model, using the relative return to a benchmark. This model makes no restriction on free-risk securities, equilibrium conditions, arbitrage limitations, utility functions, diffusion processes, or probability distributions. It considers a standard discrete-time framework, and its central assumption simply supposes that investors estimate the expected relative return to a benchmark. Its main result indicates that the asset's expected return is equal to the expected return of the lowest-risk asset (or portfolio), plus a risk premium directly proportional to the covariance between the asset's excess return and the benchmark factor.

In this paper, we extend the benchmark asset pricing model of Bergeron (2021). The extension model allows us to obtain results similar to the original model, in two distinct situations. In the first case, we assume that the asset expected relative returns are identical. In the second case, we relax this restrictive assumption and suppose that these expected values could be unequal. This suggests that a benchmark asset pricing model can be built in different contexts.

In Section II, we develop our extension model with identical expected relative returns. In Section III, we develop our model in the case where the expected relative returns are not the same. In section IV, we conclude.

II. The model with identical expected relative returns

In this section, we start by presenting our assumptions. Then, we derive the corresponding risk-return relationship.

The assumptions

Following Bergeron (2021), the link between the *Absolute Return*, *Benchmark Return*, and *Relative Return*, is given by

1 + *Absolute Return* = (1 + *Benchmark Return*) x (1 + *Relative Return*).

Therefore, we have

1 + *Relative Return* = (1 + *Absolute Return*) / (1 + *Benchmark Return*).

In the first case, the extension model is based on the definitions above, and on these assumptions:

A1 In the economy, there are many different assets and distinct investors;

A2 Investors prefer more rather than less, and are risk averse;

A3 For each asset, investors calculate the expected relative return to the benchmark;

A4 The asset expected relative returns are identical.

Assumptions A1 to A3 were already mentioned in the original framework. The new assumption A4 is introduced here to simplify the model derivation, and to show that the model's main conclusions can be obtained in different contexts.

Below, we will use these four assumptions to characterize the relationship between the risk of an asset and its expected return.

The risk-return relationship

Given the available information at time t, we suppose that investor k ($k = 1,2, \dots, K$) estimates the asset's expected relative return to the benchmark, as shown below:

$$1 + \mu_{itk} = E_{tk}\left[(1 + \tilde{R}^k_{i,t+1})/(1 + \tilde{R}^k_{b,t+1})\right], \tag{1}$$

where $\tilde{R}^k_{i,t+1}$ is the return of asset i ($i = 1, 2, \dots, N$) at time $t + 1$, for investor k, $\tilde{R}^k_{b,t+1}$ is the return of the benchmark portfolio b, at time $t + 1$, for investor k, and μ_{itk} is the corresponding expected relative return.[1] In order to simplify the notation, we can also ignore the letter k, and suppose that the representative investor estimates the following mathematical expectation:

$$1 + \mu_{it} = E_t\left[(1 + \tilde{R}_{i,t+1})/(1 + \tilde{R}_{b,t+1})\right]. \tag{2}$$

For the lowest-risk asset (or portfolio) identified by l, we have

$$1 + \mu_{lt} = E_t\left[(1 + \tilde{R}_{l,t+1})/(1 + \tilde{R}_{b,t+1})\right], \tag{3}$$

and for the benchmark portfolio, we get

$$1 + \mu_{bt} = E_t\left[(1 + \tilde{R}_{b,t+1})/(1 + \tilde{R}_{b,t+1})\right], \tag{4}$$

where, by definition, μ_{bt} corresponds to zero. In accordance with the equivalence assumption (A4), we can see that: $\mu_{it} = \mu_{lt} = \mu_{bt} = 0$. Thus, equation (2) minus (3), indicates

$$0 = E_t\left[(\tilde{R}_{i,t+1} - \tilde{R}_{l,t+1})/(1 + \tilde{R}_{b,t+1})\right]. \tag{5a}$$

Using a compact formulation, we can write

$$0 = E_t\left[\tilde{F}_{t+1}\tilde{r}_{i,t+1}\right], \tag{5b}$$

where $\tilde{F}_{t+1} \equiv (1 + \tilde{R}_{b,t+1})^{-1}$ represents the benchmark factor at time $t + 1$, and $\tilde{r}_{i,t+1} \equiv \tilde{R}_{i,t+1} - \tilde{R}_{l,t+1}$ is the excess return of asset i, at time $t + 1$. From equation 5b, the covariance implies that

[1] In this manuscript, the tilde (~) indicates a random variable. Operators E_t, V_t, and Cov_t refer respectively to mathematical expectations, variance and covariance, where index t implies that we consider the available information at time t (index k refers to investor k).

$$0 = Cov_t[\tilde{F}_{t+1}, \tilde{r}_{i,t+1}] + E_t[\tilde{F}_{t+1}]E_t[\tilde{r}_{i,t+1}]. \quad (6)$$

Isolating the excess return of the asset, we have

$$E_t[\tilde{r}_{i,t+1}] = -(1/E_t[\tilde{F}_{t+1}])Cov_t[\tilde{F}_{t+1}, \tilde{r}_{i,t+1}], \quad (7)$$

and for the benchmark portfolio, we get

$$E_t[\tilde{r}_{b,t+1}] = -(1/E_t[\tilde{F}_{t+1}])Cov_t[\tilde{F}_{t+1}, \tilde{r}_{b,t+1}]. \quad (8)$$

Introducing equation (8) in (7) gives

$$E_t[\tilde{r}_{i,t+1}] = E_t[\tilde{r}_{b,t+1}]Cov_t[\tilde{F}_{t+1}, \tilde{r}_{i,t+1}]/Cov_t[\tilde{F}_{t+1}, \tilde{r}_{b,t+1}]. \quad (9)$$

Using our definition of the excess return yields

$$E_t[\tilde{R}_{i,t+1}] = E_t[\tilde{R}_{l,t+1}] + \left(E_t[\tilde{R}_{b,t+1}] - E_t[\tilde{R}_{l,t+1}]\right) \frac{Cov_t[\tilde{F}_{t+1}, \tilde{r}_{i,t+1}]}{Cov_t[\tilde{F}_{t+1}, \tilde{r}_{b,t+1}]}. \quad (10)$$

In this manner, we can state

$$E_t[\tilde{R}_{i,t+1}] = E_t[\tilde{R}_{l,t+1}] + \lambda_t \mathrm{B}_{Bit}, \quad (11)$$

$$\lambda_t \equiv E_t[\tilde{R}_{b,t+1}] - E_t[\tilde{R}_{l,t+1}],$$

$$\mathrm{B}_{Bit} \equiv Cov_t[\tilde{F}_{t+1}, \tilde{R}_{i,t+1} - \tilde{R}_{l,t+1}]/Cov_t[\tilde{F}_{t+1}, \tilde{R}_{b,t+1} - \tilde{R}_{l,t+1}].$$

Equation (11) represents our first result. In the case where the asset expected relative returns are identical, this equation indicates that the expected return of an asset is equal to the expected return of the lowest-risk asset (or portfolio), plus a risk premium directly proportional to a benchmark beta, obtained from the covariance between the asset's excess return and the benchmark factor. Here, assumption A2 (*risk aversion*) suggests that the *parameter lambda* (λ_t), in equation (11), is positive and represents the *price of risk*, where the *quantity of risk* is measured by the *benchmark beta* (B_{Bit}).

In short, the above development demonstrates that the relationship between the risk of an asset and its expected return can be characterized using only three unrestrictive assumptions (A1, A2, and A3), plus an additional assumption (A4) on the equivalency between the expected relative return of assets.

From a theoretical point of view, our assumption A4 can be justified by the no-arbitrage principle. Indeed, according to this fundamental principle, in a complete market, we know that

$$1 = E_t\left[\widetilde{M}_{t+1}(1+\tilde{R}_{i,t+1})\right],$$

where $\widetilde{M}_{t+1}$ represents the *stochastic discount factor* at time $t+1$ (see Campbell, 2018, Chapter 4). Therefore, if we accept the no-arbitrage principle and our assumptions A1 to A3, and if we postulate that the *benchmark factor* just corresponds to the familiar *stochastic discount factor* ($\tilde{F}_{t+1} = \widetilde{M}_{t+1}$), then we can easily admit our assumption A4.

Assumption A4 can also be justified without any references to the *stochastic discount factor* or the no-arbitrage principle. In fact, assumption A4 simply suggests that sometimes, for different states of nature and probabilities, the relative return of an asset is superior to its expected value, given by the benchmark, and sometimes it is inferior. In other words, assumption A4 simply proposes that, on average, the asset's relative return is equivalent to the global central reference point (the benchmark). Nevertheless, in the next section, we will relax assumption A4.

III. The model with different expected relative returns

In this section, we develop our model with different expected relative returns. Our model development integrates an infinitesimal quantity (denoted by the letter epsilon, ε), and derives an approximate relationship.

Without the equivalency assumption (A4), equation (2) minus (3) now indicates that

$$\phi_{it} = E_t\left[(\tilde{R}_{i,t+1} - \tilde{R}_{l,t+1})/(1+\tilde{R}_{b,t+1})\right], \tag{12a}$$

with $\phi_{it} \equiv \mu_{it} - \mu_{lt}$. Using the same compact formulation as before, we can write

$$\phi_{it} = E_t\left[\tilde{F}_{t+1}\tilde{r}_{i,t+1}\right]. \tag{12b}$$

From equation 12b, the covariance implies that

$$\phi_{it} = Cov_t\left[\tilde{F}_{t+1}, \tilde{r}_{i,t+1}\right] + E_t[\tilde{F}_{t+1}]E_t\left[\tilde{r}_{i,t+1}\right]. \tag{13}$$

Integrating (13) in (12b), we get

$$1 = E_t\left[\tilde{F}_{t+1}\tilde{r}_{i,t+1}/\phi_{it}\right] = E_t\left[\tilde{F}_{t+1}\tilde{Y}_{i,t+1}\right], \tag{14}$$

where $\tilde{Y}_{i,t+1} \equiv \frac{\tilde{r}_{i,t+1}}{Cov_t[\tilde{F}_{t+1},\tilde{r}_{i,t+1}]+E_t[\tilde{F}_{t+1}]E_t[\tilde{r}_{i,t+1}]}$. For the benchmark portfolio, we have

$$1 = E_t\left[\tilde{F}_{t+1}\tilde{Y}_{b,t+1}\right]. \tag{15}$$

Thus, equation (14) minus (15) indicates

$$0 = E_t\left[\tilde{F}_{t+1}(\tilde{Y}_{i,t+1} - \tilde{Y}_{b,t+1})\right], \tag{16}$$

and the mathematical definition of covariance shows that

$$Cov_t\left[\tilde{F}_{t+1}, \tilde{Y}_{i,t+1} - \tilde{Y}_{b,t+1}\right] = -E_t[\tilde{F}_{t+1}]E_t\left[\tilde{Y}_{i,t+1} - \tilde{Y}_{b,t+1}\right], \tag{17}$$

or after simple manipulations

$$Cov_t\left[\tilde{F}_{t+1}, \tilde{Y}_{i,t+1} - \tilde{Y}_{b,t+1}\right] = E_t[\tilde{F}_{t+1}]E_t\left[\tilde{Y}_{b,t+1}\right] - E_t[\tilde{F}_{t+1}]E_t\left[\tilde{Y}_{i,t+1}\right]. \tag{18}$$

Isolating the expected value of $\tilde{Y}_{i,t+1}$, we get

$$E_t\left[\tilde{Y}_{i,t+1}\right] = E_t\left[\tilde{Y}_{b,t+1}\right] - Cov_t\left[\tilde{F}_{t+1}, \tilde{Y}_{i,t+1} - \tilde{Y}_{b,t+1}\right]/E_t[\tilde{F}_{t+1}]. \tag{19}$$

Using the properties of covariance yields

$$E_t\left[\tilde{Y}_{i,t+1}\right] = E_t\left[\tilde{Y}_{b,t+1}\right] + Cov_t\left[\tilde{F}_{t+1}, \tilde{Y}_{b,t+1}\right]/E_t[\tilde{F}_{t+1}] - Cov_t\left[\tilde{F}_{t+1}, \tilde{Y}_{i,t+1}\right]/E_t[\tilde{F}_{t+1}]. \tag{20}$$

Multiplying by the denominator of variable $\tilde{Y}_{i,t+1}$ on each side allows us to write

$$E_t\left[\tilde{r}_{i,t+1}\right] = E_t\left[\tilde{Y}_{b,t+1}\right](Cov_t\left[\tilde{F}_{t+1}, \tilde{r}_{i,t+1}\right] + E_t[\tilde{F}_{t+1}]E_t\left[\tilde{r}_{i,t+1}\right]) + Cov_t\left[\tilde{F}_{t+1}, \tilde{Y}_{b,t+1}\right](Cov_t\left[\tilde{F}_{t+1}, \tilde{r}_{i,t+1}\right] + E_t[\tilde{F}_{t+1}]E_t\left[\tilde{r}_{i,t+1}\right])/E_t[\tilde{F}_{t+1}] - Cov_t\left[\tilde{F}_{t+1}, \tilde{r}_{i,t+1}\right]/E_t[\tilde{F}_{t+1}]. \tag{21}$$

Developing, we can also write

$$E_t\left[\tilde{r}_{i,t+1}\right] = E_t\left[\tilde{Y}_{b,t+1}\right]Cov_t\left[\tilde{F}_{t+1}, \tilde{r}_{i,t+1}\right] + E_t\left[\tilde{Y}_{b,t+1}\right]E_t[\tilde{F}_{t+1}]E_t\left[\tilde{r}_{i,t+1}\right] + Cov_t\left[\tilde{F}_{t+1}, \tilde{Y}_{b,t+1}\right]Cov_t\left[\tilde{F}_{t+1}, \tilde{r}_{i,t+1}\right]/E_t[\tilde{F}_{t+1}] + Cov_t\left[\tilde{F}_{t+1}, \tilde{Y}_{b,t+1}\right]E_t\left[\tilde{r}_{i,t+1}\right] - Cov_t\left[\tilde{F}_{t+1}, \tilde{r}_{i,t+1}\right]/E_t[\tilde{F}_{t+1}]. \tag{22}$$

Regrouping the elements of (22) shows

$$E_t[\tilde{r}_{i,t+1}] = (E_t[\tilde{Y}_{b,t+1}]E_t[\tilde{F}_{t+1}] + Cov_t[\tilde{F}_{t+1}, \tilde{Y}_{b,t+1}])E_t[\tilde{r}_{i,t+1}] + \\ (E_t[\tilde{Y}_{b,t+1}] + Cov_t[\tilde{F}_{t+1}, \tilde{Y}_{b,t+1}]/E_t[\tilde{F}_{t+1}] - 1/E_t[\tilde{F}_{t+1}])Cov_t[\tilde{F}_{t+1}, \tilde{r}_{i,t+1}], \quad (23)$$

or, if you prefer

$$E_t[\tilde{r}_{i,t+1}](1 - E_t[\tilde{Y}_{b,t+1}]E_t[\tilde{F}_{t+1}] - Cov_t[\tilde{F}_{t+1}, \tilde{Y}_{b,t+1}]) = \\ (E_t[\tilde{Y}_{b,t+1}] + (Cov_t[\tilde{F}_{t+1}, \tilde{Y}_{b,t+1}] - 1)/E_t[\tilde{F}_{t+1}])Cov_t[\tilde{F}_{t+1}, \tilde{r}_{i,t+1}]. \quad (24)$$

Multiplying by $E_t[\tilde{F}_{t+1}]$ on each side of (24) implies that

$$E_t[\tilde{r}_{i,t+1}](1 - E_t[\tilde{Y}_{b,t+1}]E_t[\tilde{F}_{t+1}] - Cov_t[\tilde{F}_{t+1}, \tilde{Y}_{b,t+1}])E_t[\tilde{F}_{t+1}] = \\ (E_t[\tilde{Y}_{b,t+1}]E_t[\tilde{F}_{t+1}] + Cov_t[\tilde{F}_{t+1}, \tilde{Y}_{b,t+1}] - 1)Cov_t[\tilde{F}_{t+1}, \tilde{r}_{i,t+1}]. \quad (25)$$

Multiplying by (-1) on each side of (25) also implies that

$$E_t[\tilde{r}_{i,t+1}](1 - E_t[\tilde{Y}_{b,t+1}]E_t[\tilde{F}_{t+1}] - Cov_t[\tilde{F}_{t+1}, \tilde{Y}_{b,t+1}])E_t[\tilde{F}_{t+1}](-1) = \\ (1 - E_t[\tilde{Y}_{b,t+1}]E_t[\tilde{F}_{t+1}] - Cov_t[\tilde{F}_{t+1}, \tilde{Y}_{b,t+1}])Cov_t[\tilde{F}_{t+1}, \tilde{r}_{i,t+1}]. \quad (26a)$$

To simplify the notation, we can write

$$E_t[\tilde{r}_{i,t+1}](A_t)E_t[\tilde{F}_{t+1}](-1) = (A_t)Cov_t[\tilde{F}_{t+1}, \tilde{r}_{i,t+1}], \quad (26b)$$

with $A_t \equiv 1 - E_t[\tilde{Y}_{b,t+1}]E_t[\tilde{F}_{t+1}] - Cov_t[\tilde{F}_{t+1}, \tilde{Y}_{b,t+1}]$. From equation (15), and the covariance definition, it is easy to see that A_t is equal to zero. Nevertheless, to obtain an approximation of the risk-return relationship we can integrate the infinitesimal value ε into equation (26b), and postulate that

$$E_t[\tilde{r}_{i,t+1}](A_t + \varepsilon)E_t[\tilde{F}_{t+1}](-1) \approx (A_t + \varepsilon)Cov_t[\tilde{F}_{t+1}, \tilde{r}_{i,t+1}], \quad (27)$$

where the symbol ≈ means *approximately equal to*. Dividing by $(A_t + \varepsilon)$ on each side of equation (27), and isolating the expected excess return of the asset, we have

$$E_t[\tilde{r}_{i,t+1}] \approx -(1/E_t[\tilde{F}_{t+1}])Cov_t[\tilde{F}_{t+1}, \tilde{r}_{i,t+1}]. \quad (28)$$

For the benchmark portfolio, we also have

$$E_t[\tilde{r}_{b,t+1}] \approx -(1/E_t[\tilde{F}_{t+1}])Cov_t[\tilde{F}_{t+1}, \tilde{r}_{b,t+1}]. \quad (29)$$

In this manner, after simple manipulations (see equations 9 and 10), we obtain the following expression

$$E_t\left[\tilde{R}_{i,t+1}\right] \approx E_t\left[\tilde{R}_{l,t+1}\right] + \lambda_t \mathrm{B}_{Bit}. \tag{30}$$

Equation (30) represents our second result. In the case where the asset expected relative returns are distinct, this equation indicates that the expected return of an asset is *approximately* equal to the expected return of the lowest-risk asset (or portfolio), plus a risk premium directly proportional to the benchmark beta. Ignoring the approximation, we can see that the risk-return relationship obtained here is identical to our previous result, expressed by equation (11).

IV. Conclusion

In this working paper, we used two distinct contexts to demonstrate that the expected return of an asset is equal to the expected return of the lowest-risk asset, plus a risk premium directly proportional to the covariance between the asset's excess return and the benchmark factor. In the first case, we assumed that the asset expected relative returns are the same. In the second case, we relaxed this restrictive assumption, and supposed that these expected values could be unequal. Overall, this suggests that a risk-return relationship based on a *benchmark appr*oach can be derived in different contexts, with or without classical arbitrage restrictions.

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