Monotonicity of certain left and right Riemann sums

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Abstract. In an otherwise instructive 2012 article, Szilárd András provided a flawed argument purportedly establishing that the left (resp. right) Riemann sum of $f(x) = \frac{1}{1+x^2}$ with respect to the uniform partition of $[0, 1]$ into n equal intervals is monotonically decreasing (resp. increasing) relative to n. A few years later, D. Borwein, J. M. Borwein and B. Sims developed a symmetrization technique that allowed them to provide a rectified proof that the right Riemann sum of $f(x) = \frac{1}{1+x^2}$ really is monotonically increasing relative to n. They also provided numerical evidence suggesting that the left Riemann sum is decreasing but they did not succeed in proving it. In the first part of this paper, we exploit the symmetrization technique to provide a proof that the left Riemann sum is indeed decreasing with respect to n. Subsequently, we show, using elementary calculus techniques, some trigonometry computations as well as calculations involving generalized binomial coefficients, that the left and right Riemann sums with respect to the uniform partition of $[0, 1]$ of the family of functions of the form $\sin^p(\pi x)$ are monotonically increasing relative to n, regardless of the value of $p \in (0, 2)$. In so doing, we answer a problem that came up in the context of foundational research on questions situated at the intersection of matrix theory and metric geometry,

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1. Introduction

The concept of "infinite series" emerged in Ancient Greece in the late Classical period. Such series were inherent in the method of exhaustion which uses an infinite sequence of approximations for finding lengths of curves, areas of shapes bounded by the simple curves and volumes of simple solid bodies. This method

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was developed in antiquity by Greek mathematicians such as Antiphon of Rhamnus (c. 480 BC – 411 BC) [52], Democritus of Abdera (c. 460 BC – c. 370 BC) [53], and Eudoxus of Cnidus (c. 400 BC – c. 350 BC) [54]. The idea that a non-terminating succession of addition of quantities to a given starting quantity could bring about a finite result remained however controversial for quite some time. This is evidenced by the well-known philosophical paradox of Achilles and the tortoise devised by the Greek philosopher Zeno of Elea (c. 495 BC – c. 430 BC) to illustrate some counter-intuitive properties of infinite sums and limiting processes. It is Archimedes of Syracuse (c. 287 BC – 212 BC), in the Hellenistic period, who produced the first known summation of an infinite series [6].

According to Godfrey Harold Hardy [41, p. 1], the concept of convergence was very much familiar to great mathematicians of the sixteenth, seventeenth and eighteenth centuries of our era such as Gerardus Mercator (1512–1594), Grégoire de Saint-Vincent (1584–1667) who gave the first definition of limit of a geometric series in 1647 in his *Opus Geometricum* [29], Bonaventura Cavalieri (1598–1647), John Wallis (1616–1703), William Brouncker (1620–1684), Isaac Barrow (1630–1677), and James Gregory (1638–1675) who is credited with coining the word 'convergent' in 1667 in *Vera Circuli et Hyperbalae Quadrature* [40]. Be that as it may, the manipulation of infinite series remained more of an art than a science since mathematicians, in this day and age, "were handicapped by the lack of serviceable criteria for convergence" [41, p. 13].

Sir Isaac Newton (1642–1727), who made an extensive use of infinite series in solving quadrature problems [41, p. 13], was the first mathematician to properly master a powerful method of testing for the convergence or divergence of some infinite series [57].

Prior to Leonhard Euler (1707–1783), with the notable exception of certain passages in the correspondance [50, 51] of Gottfried Wilhelm Leibniz (1646–1716) and Johann Bernoulli (1667–1748) that were metaphysical in nature and of an account by the Italian mathematician and priest Guido Grandi (1671–1742) in a 1703 book with a deep religious overtone [39] of a series that is now named after him, divergent series were mostly excluded from the scope of the mathematical investigation on infinite series. Disinclined to provide a definition of what is meant by the sum of a divergent series, Euler nevertheless committed himself to defend the principle that there is only one "natural" sum which it is "reasonable" to assign to a divergent series (see for instance [32]).

Consider Grandi's series

$$G := \sum_{n=1}^{\infty} a_n \;=\; 1 - 1 + 1 - 1 + \cdots ,$$

whose k-th partial sum is

$$s_k \;=\; \sum_{n=1}^{k} a_n = a_1 + a_2 + \cdots + a_k \;=\; \begin{cases} 1, & \text{for odd } k, \\ 0, & \text{for even } k. \end{cases}$$

Although it has accumulation points at 0 and 1, the associated sequence of partial sums of Grandi's series clearly does not approach any value. Therefore, Grandi's series is divergent. We can nevertheless devise arguments leading us to attribute some plausible meaning to such an expression. The simplest such argument involves some simple algebraic manipulations. Set aside for a moment the pivotal question of what the sum of a divergent series actually means and play along with the following symbolic game:

$$G = 1 - 1 + 1 - 1 + \cdots \;=\; 1 - (1 - 1 + 1 - 1 + \cdots) = 1 - G,$$

and thus $G = \frac{1}{2}$. This flawed and logically unsound argument seems to point out that Grandi's series sum up to $\frac{1}{2}$. We shall come back to this idea a little later.

At the end of the eighteenth century, Joseph-Louis Lagrange (1736-1813) denounced in his *Théorie des fonctions analytiques* [49] the lack of rigor in contemporary mathematics resulting in frequent appearances of paradoxes and contradictions. He urged the mathematical community to seek secure foundations for calculus and warned that failure to do so would seriously hamper further development. Establishing mathematical analysis on more solid footing occupied mathematicians for much of the following century.

In his 1821 seminal textbook in infinitesimal calculus entitled *Cours d'Analyse de l'École Royale Polytechnique; I.^{re} Partie. Analyse algébrique* , Augustin Louis Cauchy (1789–1857) formalized ideas concerning convergence and divergence of sequences whose elements become arbitrarily close to each other as the sequence progresses. Moreover, he clearly and explicitly defined the "natural sum" of an infinite series based on the concept of limit, and proclaimed the banishment of divergent series from the domain of rigorous mathematics. Following the publication of Cauchy's masterpiece, the discovery and development of effective convergence criteria began. At the same time, analysis moved steadfastly away from series which did not converge. For more than half a century, only a few bold mathematicians such as Jean-Baptiste Joseph Fourier (1768–1830) [36], Siméon Denis Poisson (1781–1840) [60], and Niels Henrik Abel (1802–1829) [1] dared venture away from the prevailing orthodoxy [41, p. 17]. Fourier, for instance, noted that the calculation based on certain non-convergent series (in the sense of Cauchy) that come up naturally in astronomy were valid and verifiable by other means.

The interest in the study of divergent series reawaken after Ernesto Cesàro (1859-1906) devised a method to assign values to some infinite series that are not convergent in the sense of Cauchy. His method, which had been implicitly used by Georg Frobenius in 1880 [37] and Otto Hölder in 1882 [46], also assigned convergent series the same value as that assigned by Cauchy's concept of convergence.

We shall illustrate Cesàro's method by means of an example. Consider once more Grandi's series

$$G := \sum_{n=0}^{\infty} a_n \;=\; 1 - 1 + 1 - 1 + \cdots ,$$

and its associated sequence of partial sums $(s_k)_{n=1}^{\infty} = (1,0,1,0,\dots)$. Let $(t_n)_{n=1}^{\infty}$ be the sequence of arithmetic means of the first n partial sums of G:

$$t_j \;=\; \frac{1}{j}\sum_{k=1}^{j} s_k \;=\; \left(1,\frac{1}{2},\frac{2}{3},\frac{1}{2},\frac{3}{5},\frac{1}{2},\frac{4}{7},\dots\right) \;=\; \begin{cases} \frac{1}{2}+\frac{1}{2j}, & \text{for odd } j,\\ \frac{1}{2}, & \text{for even } j, \end{cases}$$

thus proving that $\lim_{j\to\infty} t_j = \frac{1}{2}$. In modern parlance, we say that Grandi's series converges to $\frac{1}{2}$ in the sense of Cesàro.

In a now-famous paper published in 1890 [27], Cesàro explicitly formulated a "theory of divergent series" when he introduced a whole family of higher-order summability methods which have since been called (C,α) for non-negative integers α, with $(C,0)$-summation being just ordinary summation and $(C,1)$-summation being the Cesàro summation outlined above. This moment in time marks the birth of *summability theory* (i.e., the theory of the assigning of a value to an infinite series) as it was soon revealed that summability methods were useful in practice and often led to momentous results (see for instance [34, 35]).

By the end of the nineteenth century, several alternative summability methods of assigning a value to infinite series by transforming a given sequence of partial sums into another sequence on which Cauchy's method is applicable were invented, improved or extended by dozens of eminent mathematicians like Stefan Banach [4], Sidney Chapman [28, 42], John Edensor Littlewood [43], Godfrey Harold Hardy [41, 42, 44], Konrad Knopp [47], Ervand George Kogbetliantz [48], Niels Erik Nørlund [59], Marcel Riesz [44, 61], Issai Schur [62], Alfred Tauber [65], Otto Toepliz [66], Georgy Feodosevich Voronoy [69], Norbert Wiener [73], and Antoni Zygmund [74]. For a survey of those methods and many more, see [30]. See also G. H. Hardy's *Divergent Series*. This book, the last one that he penned, is widely considered to be the classic reference in the field.

Summability theory has remained an active area of research in recent years and it has found many applications in various branches of analysis such as harmonic analysis, functional analysis, operator theory, approximation theory, fixed-point theory, and complex analysis. But nowhere is the usefulness and fruitfulness of summability methods more evident than in the theory of function spaces, including Hardy spaces [71, 70, 72], (weighted) Dirichlet spaces [55, 67], Bergman space [5], de Branges-Rovnyak spaces [2, 31], and Besov spaces [33, 68], just to name a few.

2. Background and context

In this section, we present the state of knowledge in a somewhat underdeveloped subsidiary branch of summability theory that is relevant to our main purpose, that is the identification of criteria under which certain Riemann sums of integrable real-valued functions monotonically converge to a limit. In passing, we provide an answer to an open question of David Borwein (a well-known summabilist, see

[8, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]), Jonathan M. Borwein and Brailey Sims.

The *Riemann sum* of a bounded function $f : [a, b] \to \mathbb{R}$ over a partition

$$a = x_0 < \ x_1 < \cdots < x_{n-1} < x_n = b$$

is a finite sum of the form

$$\sum_{k=1}^{n} f(x_k^*)(x_k - x_{k-1}),$$

where for each value of k the element x_k^* is chosen arbitrarily in the subinterval $[x_{k-1}, x_k]$. If f is Riemann integrable, then the difference between the Riemann sum and the Riemann integral $\int_a^b f(x)\,dx$ approaches zero as n tends to infinity, no matter which x_k^*'s were chosen.

When x_k^* is set to be x_{k-1}, the left endpoint of the subinterval $[x_{k-1}, x_k]$, for all k, it is customary to speak of the *left Riemann sum*. As for when x_k^* is set to be x_k, the right endpoint of the subinterval $[x_{k-1}, x_k]$, for all k, we speak of the *right Riemann sum*. When f is decreasing on the interval $[a, b]$, the left Riemann sum gives an overestimate of the integral, and the right Riemann sum gives an underestimate. The opposite is true is when the function is increasing.

In what follows, we shall restrict ourselves to Riemann integrable functions defined on the interval $[0, 1]$. We shall denote the *left Riemann sum of f with respect to the uniform partition $\mathcal{U}_n$ of* $[0, 1]$ into n intervals of equal width by

$$L_n(f) := \ \frac{1}{n} \sum_{k=0}^{n-1} f\left(\tfrac{k}{n}\right).$$

Similarly, we shall designate the *right Riemann sum of f with respect to the uniform partition $\mathcal{U}_n$ of* $[0, 1]$ by

$$R_n(f) := \ \frac{1}{n} \sum_{k=1}^{n} f\left(\tfrac{k}{n}\right).$$

In [3], Szilárd András recounts how the following two problems arose in a group discussion during an Inquiry Based Learning (IBL) teacher training session focused on the introduction of the Riemann integral based on some real world problems:

Problem 1. Prove that $L_n(\frac{1}{1+x^2}) = \frac{1}{n} \sum\limits_{k=0}^{n-1} \frac{1}{1+\left(\frac{k}{n}\right)^2} = \sum\limits_{k=0}^{n-1} \frac{n}{n^2+k^2}$ decreases monotonically to $\int_0^1 \frac{1}{1+x^2}\,dx$ as n tends to infinity.

Problem 2. Prove that $R_n(\frac{1}{1+x^2}) = \frac{1}{n} \sum\limits_{k=1}^{n} \frac{1}{1+\left(\frac{k}{n}\right)^2} = \sum\limits_{k=1}^{n} \frac{n}{n^2+k^2}$ increases monotonically to $\int_0^1 \frac{1}{1+x^2}\,dx$ as n tends to infinity.

Departing for a moment from the specific case of the function $f(x) = \frac{1}{1+x^2}$, the author provides the following set of sufficient conditions on $f : [0,1] \to \mathbb{R}$ that ensure the monotonicity of $L_n(f)$ and $R_n(f)$ as functions of n.

Theorem 3. [3, Theorem 1] *If $f : [0,1] \to \mathbb{R}$ is convex and decreasing on the interval $[0,1]$, then $L_n(f)$ decreases monotonically and $R_n(f)$ increases monotonically relative to n.*

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The author then mistakenly asserts that Theorem 4 applies to the function $f(x) = \frac{1}{1+x^2}$, which is not the case since $f(x)$ has an inflection point at $1/\sqrt{3}$.

Ironically, the blunder of Szilárd András (in an otherwise interesting paper) turned out to be fertile in mathematical ideas as it caught the attention of Jonathan Michael Borwein. In the process of producing [25], the late Director of the Centre for Computer-Assisted Research Mathematics and its Applications at the University of Newcastle, got interested in the behavior of the sequences $\sigma_n := \sum_{k=0}^{n-1} \frac{n}{n^2+k^2}$, and $\tau_n := \sum_{k=1}^{n} \frac{n}{n^2+k^2}$. Together with his late father, David Borwein, and longtime collaborator Brailey Sims, Jonathan M. Borwein penned an article [13] in which new conditions ensuring the monotonicity of some left and right Riemann sums are provided. This paper turned out the be one of the last on which Jonathan M. Borwein worked on before his sudden and unexpected passing on August 2, 2016.

Here is a summary of some of the highlights of their paper. First, they provide the following slight extension to Theorem 3.

Theorem 5. [13, Theorem 3] *If the function $f : [0,1] \to \mathbb{R}$ is convex on the interval $[0,c]$ for some $0 < c < 1$, concave on $[c,1]$, and decreasing on $[0,1]$, then $R_n(f)$ increases monotonically and $L_n(f)$ decreases monotonically relative to n.*

One might expect that Theorem 4 admits a similar extension. But it suffices to consider the characteristic function of the interval $\left[0, \frac{1}{2}\right]$ to see that such is not the case. However, by applying Theorem 5 to $-f$, we obtain the following result.

Corollary 6. [13, Theorem 4] *If the function $f : [0,1] \to \mathbb{R}$ is concave on the interval $[0,c]$ for some $0 < c < 1$, convex on $[c,1]$, and increasing on $[0,1]$, then $R_n(f)$ decreases monotonically and $L_n(f)$ increases monotonically relative to n.*

Exploiting the same method of proof, the authors also derive the following result which we will use later on.

Theorem 7. [13, Theorem 5] *If the function $f : [0,1] \to \mathbb{R}$ is concave on the interval $[0,1]$, with maximum $f(c)$ for some $0 < c < 1$, then $R_n(f) - \frac{f(c)-f(0)}{n}$ increases monotonically relative to n.*

The core of the paper is devoted to analyzing how the behavior of the *symmetrization* of a given function affect the behavior of its left and right Riemann sums. Given a function $f : [0, 1] \to \mathbb{R}$, its *symmetrization* (about $x = \frac{1}{2}$) is defined to be

$$\mathcal{F}_{1/2}(x) := \mathcal{F}(x) = \frac{f(x) + f(1-x)}{2}.$$

One can easily check that the symmetrization of a convex (resp. concave) function is again convex (resp. concave). Interestingly, as noted by Borwein *et al.* [13], though the symmetrization process cannot destroy convexity or concavity, it can generate either of these properties. Such is the case of the function $f(x) = \frac{1}{1+x^2}$. As can be verified by elementary calculus, this function – which is neither convex on the whole of $[0, 1]$ nor concave – nonetheless admits a concave symmetrization.

The following result shows how the symmetrization of a function relates with the monotonic behavior of its left and right Riemann sums.

Theorem 8. [13, Corollary 2] *If $f : [0, 1] \to \mathbb{R}$ has a concave symmetrization and verifies $f(0) > f(\frac{1}{2})$, then $R_n(f)$ increases monotonically relative to n.*

Observing that $R_n(f(1-x)) = L_n(f(x))$, and thus we obtain, by applying Theorem 8 to $-f(x)$, $f(1-x)$, and $-f(1-x)$ respectively, the following corollaries:

Corollary 9. *If f has a convex symmetrization and verifies $f(0) < f(\frac{1}{2})$, then $R_n(f)$ decreases monotonically relative to n.*

Corollary 10. *If f has a concave symmetrization and verifies $f(\frac{1}{2}) < f(1)$, then $L_n(f)$ increases monotonically relative to n.*

Corollary 11. *If f has a convex symmetrization and verifies $f(\frac{1}{2}) > f(1)$, then $L_n(f)$ decreases monotonically relative to n.*

It is worth noting that since the symmetrization of f never destroy convexity or concavity, Theorem 8 and Corollary 11 respectively extend Theorems 4 and 3. Using Theorem 8, Borwein *et al.* provide a rectified proof that

$$R_n\left(\tfrac{1}{1+x^2}\right) = \sum_{k=1}^{n} \frac{n}{n^2 + k^2}$$

increases monotonically to $\int_0^1 \frac{1}{1+x^2}\, dx = \frac{\pi}{4}$ as n tends to infinity [13, Example 3]. Somewhat surprisingly, although it is straightforward to check through numerical experiments, the sums

$$L_n\left(\tfrac{1}{1+x^2}\right) = \sum_{k=0}^{n-1} \frac{n}{n^2 + k^2}$$

exhibit an apparent monotonic decreasing behavior, none of the theorems and corollaries stated above supply a straightforward rigorous proof. We establish this result.

3. The left Riemann sum of $\frac{1}{1+x^2}$

In this Section, we finalize the process of sealing off the breach in Szilárd András' 2012 article by showing that the left Riemann of $\frac{1}{1+x^2}$ with respect to the uniform partition of $[0,1]$ is indeed monotonically decreasing relative to n.

First, we define $f, g, h : [0,1] \to \mathbb{R}$ by:

$$\begin{aligned} f(x) &:= \frac{1}{1+x^2}, \\ g(x) &:= \begin{cases} 0, & \text{if } 0 \le x < \frac{1}{2}, \\ \left(x - \frac{1}{2}\right)^2, & \text{if } \frac{1}{2} \le x \le 1, \end{cases} \\ h(x) &:= f(x) + g(x). \end{aligned}$$

Claim 1: $L_n(g)$ is monotonically increasing relative to n.

Proof. This can be deduced from Corollary 6, since g is concave on the interval $\left[0, \frac{1}{2}\right]$, convex on $\left[\frac{1}{2}, 1\right]$, and increasing on $[0,1]$. □

Claim 2: $L_n(h)$ is monotonically decreasing relative to n.

Proof. The symmetrization of h is given by

$$\begin{aligned} \mathcal{H}(x) &= \frac{h(x) + h(1-x)}{2} \\ &= \frac{1}{2}\left(\frac{1}{1+x^2} + \frac{1}{1+(1-x)^2} + \left(x - \tfrac{1}{2}\right)^2\right). \end{aligned}$$

Therefore,

$$\mathcal{H}''(x) = \frac{y^6 + 24y^4 + 51y^2 + 2 - 6y(y^4 + 10y^2 + 1)}{(1+x^2)^3\left(1+(1-x)^2\right)^3},$$

where $y := x(1-x)$. Note that $0 \le y \le \frac{1}{4}$, whenever $0 \le x \le 1$. Thus,

$$\begin{aligned} \mathcal{H}''(x) &\ge \frac{y^6 + 24y^4 + 51y^2 + 2 - \frac{6}{4}(y^4 + 10y^2 + 1)}{(1+x^2)^3\left(1+(1-x)^2\right)^3} \\ &= \frac{y^6 + \frac{45}{2}y^4 + 36y^2 + \frac{1}{2}}{(1+x^2)^3\left(1+(1-x)^2\right)^3} \\ &\ge 0. \end{aligned}$$

This implies that $\mathcal{H}(x)$ is convex on $[0,1]$. Moreover, $h(1/2) = \frac{4}{5} > \frac{3}{4} = h(1)$. Hence, we deduce from Corollary 11 that $L_n(h)$ decreases monotonically relative to n. □

Observe that

$$L_n(h) = L_n(f+g) = L_n(f) + L_n(g).$$

Claim 1 and Claim 2 together imply that $L_n(f) = L_n(h) - L_n(g)$, being the sum of monotonically decreasing functions, decreases monotonically as n grows to infinity.

4. A finer analysis via experimentation

As a way of highlighting the many ways that digital experimentations can enrich mathematical research, Borwein *et al.* [13] went on to analyze the family of functions $f_b : [0,1] \to \mathbb{R}$ defined by

$$f_b(x) := \frac{1}{x^2 - bx + 1},$$

for the parameter range $|b| < 2$.

Following a careful study of f_b and of the concavity/convexity of the symmetrization of f_b, the authors conclude that $L_n(f_b)$ is monotonically decreasing relative to n for $b \in [-2, \alpha_-]$, where α_- stands for the negative root of $b^3 - 3b^2 + 3 = 0$ which is ≈ -0.8794. Moreover, they show that $R_n(f_b)$ is monotonically increasing relative to n for $b \in [(3 - \sqrt{13})/3, 1/2]$. In the remainder of this section, we shall briefly sketch how the method used to demonstrate that the Riemann of $\frac{1}{1+x^2}$ is monotonically decreasing relative to n can be further exploited to sharpen this analysis.

The left Riemann sums of $\frac{1}{1-bx+x^2}$. Remark that for $b \in (-\infty, -1]$ the function $f_b(x)$ is convex and decreasing on $[0,1]$. Therefore, Theorem 3 implies that $L_n\left(\frac{1}{1-bx+x^2}\right)$ is monotonically decreasing relative to n.

We now turn our attention on the case where $b > -1$. Consider the two-parameter function $g_{a,c} : [0,1] \to \mathbb{R}$ defined by

$$g_{a,c}(x) := \begin{cases} 0, & \text{if } 0 \le x < \frac{1}{2}, \\ a\left(x - \frac{1}{2}\right)^2 + c\left(x - \frac{1}{2}\right)^4, & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

We consider two cases.

1. Let $b \in \left(-1, \frac{3-\sqrt{13}}{4}\right)$. If $a := 1 - b^2$ and $c := 0$, then Theorem 5 implies that $L_n(g_{a,c})$ is monotonically increasing relative to n. Furthermore, one can check that the symmetrization of $h_b(x) := f_b(x) + g_{a,c}(x)$ around $\frac{1}{2}$ is convex and that $h_b(1/2) > h_b(1)$. Thus, it follows from Corollary 11 that $L_n(h_b)$ is monotonically decreasing relative to n. Hence $L_n\left(\frac{1}{1-bx+x^2}\right)$ is monotonically decreasing relative to n.
2. Let $b \in \left[\frac{3-\sqrt{13}}{4}, \alpha\right)$, where $\alpha \approx 0.493862$. If

$$a := -32\frac{4b^2 - 6b - 1}{(5 - 2b)^3}$$

and

$$c := 16\left(\frac{59 - 198b + 164b^2 - 40b^3}{(2-b)(5-2b)^3} - \varepsilon\right),$$

then Theorem 5 implies that $L_n(g_{a,c})$ is monotonically increasing relative to n. Furthermore, the symmetrization of $h_b(x) := f_b(x) + g_{a,c}(x)$ around $\frac{1}{2}$ is convex and $h_b(1/2) > h_b(1)$. Thus, it follows from Corollary 11 that $L_n(h_b)$ is monotonically decreasing relative to n. Hence $L_n\left(\frac{1}{1-bx+x^2}\right)$ is monotonically decreasing relative to n.

We described the behavior of the left Riemann sums of $\frac{1}{1-bx+x^2}$ for $b \in (-\infty, \approx 0.493862)$. Since numerical analysis suggests that $L_n(f_b)$ no longer display monotonic behavior for $b \geq 0.5$, this description is just short of being exhaustive.

The right Riemann sums of $\frac{1}{1-bx+x^2}$. For $b \in (-\infty, -1]$, we may once again apply Theorem 3 and conclude that $R_n\left(\frac{1}{1-bx+x^2}\right)$ is monotonically increasing relative to n.

As for the case where $b > -1$, we consider the three-parameter function $k_{a,c,d} : [0,1] \to \mathbb{R}$ defined by

$$k_{a,c,d}(x) := \begin{cases} a\left(x-\frac{1}{2}\right)^2 + c\left(x-\frac{1}{2}\right)^4 + d\left(x-\frac{1}{2}\right)^6, & \text{if } 0 \leq x < \frac{1}{2}, \\ 0, & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

We consider two cases.

1. Let $b \in \left(-1, \frac{3-\sqrt{13}}{4}\right)$. If $a := \frac{4}{9}(5-2\sqrt{13})$, $c := 0$ and $d := 0$, then Theorem 6 implies that $R_n(k_{a,c,d})$ is monotonically decreasing relative to n. Furthermore, one can check that the symmetrization of $h_b(x) := f_b(x) + k_{a,c,d}(x)$ around $\frac{1}{2}$ is concave and that $h_b(0) > h_b(1/2)$. Thus, it follows from Theorem 8 that $R_n(h_b)$ is monotonically decreasing relative to n. Hence, $R_n\left(\frac{1}{1-bx+x^2}\right)$ is monotonically increasing relative to n.
2. Let $b \in \left[\frac{3-\sqrt{13}}{4}, 1\right]$. If

$$a := 32\frac{1+6b-4b^2}{(5-2b)^3},$$

and

$$c := -\frac{128}{3}\frac{1+6b-4b^2}{(5-2b)},$$

and finally

$$d := \frac{512}{15}\frac{1+6b-4b^2}{(5-2b)^3},$$

then Theorem 5 implies that $R_n(k_{a,c,d})$ decreases monotonically relative to n. Furthermore, the symmetrization of $h_b(x) := f_b(x) + k_{a,c,d}(x)$ around $\frac{1}{2}$ is concave and $h_b(0) > h_b(1/2)$. Thus, it follows from Theorem 8 that $R_n(h_b)$ is monotonically increasing relative to n. Hence, $R_n\left(\frac{1}{1-bx+x^2}\right)$ is monotonically increasing relative to n.

5. Riemann sum of trigonometric functions

We now turn our attention to studying the of monotonicity of the left and right Riemann sums with respect to the uniform partition of $[0,1]$ for the specific family of functions of the form $\sin^p(\pi x)$, where $p \in (0,2]$. As a matter of fact, we may restrict our attention only to the left Riemann sum since $\sin^p(\pi x)$ is symmetric about the midpoint of $[0,1]$, so that for all n, we have

$$L_n\big(\sin^p\left(\pi(1-x)\right)\big) = R_n\big(\sin^p(\pi x)\big).$$

As simple-looking as this question may appear at first glance, it is worth mentioning that answering it allowed us to solve a problem that came up in the context of foundational research on questions situated at the intersection of matrix theory and metric geometry. Indeed, one can show (see [26]) that the diameter of the n-dimensional Birkhoff polytope, i.e., the set of doubly stochastic matrices [7], with respect to the Schatten p-norm $(1 \leq p \leq 2)$ verifies

$$\operatorname{diam}_{\mathcal{S}_p}(\mathcal{D}_n) \;\geq\; 2\left(\sum_{k=1}^{n} \sin^p\left(\tfrac{k\pi}{n}\right)\right)^{1/p}.$$

In trying to demonstrate that this inequality is in fact an identity, it was revealed that we need to find out if the right Riemann sum of $\sin^p(\pi x)$ is monotonically increasing relative to n for $p \in [1,2)$.

As a matter of fact, two cases stand apart, i.e., $p = 1$ and $p = 2$. It is within the reach of anyone familiar with complex numbers to check that the left and right Riemann sums of both $\sin(\pi x)$ and $\sin^2(\pi x)$ with respect to the uniform partition of $[0,1]$ are monotonically increasing relative to n. For $p = 1$ we have:

$$\begin{aligned}
L_n\big(\sin(\pi x)\big) &= \frac{1}{n}\sum_{k=0}^{n-1} \sin\left(\tfrac{k\pi}{n}\right)\\
&= \frac{1}{n}\Im\left(\sum_{k=0}^{n-1}\exp\left(\tfrac{ik\pi}{n}\right)\right) \;=\; \frac{1}{n}\Im\left(\frac{\exp(i\pi)-1}{\exp\left(\frac{i\pi}{n}\right)-1}\right)\\
&= \frac{1}{n}\Im\left(\frac{\exp\left(\frac{i\pi}{2}\right)\left(\exp\left(\frac{i\pi}{2}\right)-\exp\left(-\frac{i\pi}{2}\right)\right)}{\exp\left(\frac{i\pi}{2n}\right)\left(\exp\left(\frac{i\pi}{2n}\right)-\exp\left(-\frac{i\pi}{2n}\right)\right)}\right)\\
&= \frac{1}{n}\Im\left(\exp\left(\tfrac{i(n-1)\pi}{2n}\right)\frac{\sin\left(\frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{2n}\right)}\right) \;=\; \frac{1}{n}\sin\left(\tfrac{(n-1)\pi}{2n}\right)\frac{1}{\sin\left(\frac{\pi}{2n}\right)}\\
&= \frac{1}{n}\frac{\sin\left(\frac{\pi}{2}-\frac{\pi}{2n}\right)}{\sin\left(\frac{\pi}{2n}\right)} \;=\; \frac{1}{n}\frac{\cos\left(-\frac{\pi}{2n}\right)}{\sin\left(\frac{\pi}{2n}\right)} \;=\; \frac{1}{n}\cot\left(\frac{\pi}{2n}\right),
\end{aligned}$$

whereas for $p = 2$, we have $L_1\big(\sin^2(\pi x)\big) = 0$ and, for $n \geq 2$,

$$\begin{aligned} L_n\big(\sin^2(\pi x)\big) &= \frac{1}{n}\sum_{k=0}^{n-1}\sin^2\left(\tfrac{k\pi}{n}\right) = \frac{1}{n}\sum_{k=0}^{n-1}\left(\frac{1-\cos\left(\frac{2k\pi}{n}\right)}{2}\right) \\ &= \frac{1}{2} + \frac{1}{n}\sum_{k=0}^{n-1}\cos\left(\tfrac{2k\pi}{n}\right) = \frac{1}{2} - \frac{1}{n}\Re\left(\sum_{k=0}^{n-1}\exp\left(\tfrac{i2k\pi}{n}\right)\right) \\ &= \frac{1}{2} - \frac{1}{n}\Re\left(\frac{1-\exp(i2\pi)}{1-\exp\left(\frac{i2\pi}{n}\right)}\right) = \frac{1}{2}. \end{aligned}$$

In order to determine whether or not the left and right Riemann sums of $\sin^p(\pi x)$ behave similarly for $1 < p < 2$, we have to delve deeper. To gain insight into the behavior of the function $\sin^p(\pi x)$, we compute its second derivative, i.e.,

$$\pi^2 p\sin^{p-2}(\pi x)\Big((p-1)\cos^2(\pi x) - \sin^2(\pi x)\Big)$$

Thus, concavity changes when, if ever, $(p-1)\cos^2(\pi x) - \sin^2(\pi x) = 0$. Observe that

$$(p-1)\cos^2(\pi x) - \sin^2(\pi x) = 0 \iff x = \tfrac{1}{\pi}\arctan\left(\sqrt{p-1}\right).$$

This implies that, for $p \in (0,1]$, the function $\sin^p(\pi x)$ is concave on $[0,1]$, increasing on $\left[0,\frac{1}{2}\right]$, decreasing on $\left[\frac{1}{2},1\right]$, and symmetric with respect to its middle point. Hence, it follows from Borwein's aforementioned result (Theorem 7) that $L_n\big(\sin^p(\pi x)\big) - \frac{1}{n}$ is monotonically increasing relative to n.

The behavior of the function $\sin^p(\pi x)$ is more delicate for $p > 1$. It is concave on $\left[0,\frac{1}{\pi}\arctan\left(\sqrt{p-1}\right)\right]$, convex on $\left[\frac{1}{\pi}\arctan\left(\sqrt{p-1}\right),1\right]$, increasing on $[0,1/2]$ and decreasing on $[1/2,1]$.

Perhaps surprisingly, given that the function $\sin^p(\pi x)$ behaves differently with respect to convexity if $p \leq 1$ or $p > 1$, we shall prove in a single stroke that the following holds true.

Theorem 12. *Given any $p \in (0,2)$, the left and right Riemann sums of $\sin^p(\pi x)$ with respect to the uniform partition of $[0,1]$ are monotonically increasing relative to n.*

To prove this result, we need to develop some preliminary results.

6. Some prerequisites

In this section, we shall state some results that will prove instrumental in the proof of Theorem 12. We begin with a trigonometric identity.

Trigonometric Identity 13. For any nonnegative integers n, j,

$$\sum_{k=0}^{n-1}\cos^{2j}\left(\tfrac{k\pi}{n}\right) = \frac{n}{4^j}\sum_{k=-\lfloor j/n\rfloor}^{\lfloor j/n\rfloor}\binom{2j}{j+kn}.$$

Proof. See [56, eq. (3)]. □

We now turn our attention towards some binomial identities. But before doing so, we shall discuss some generalizations of the definition of the binomial coefficients. Given nonnegative integers k and n with $0 \leq k \leq n$, the binomial coefficient $\binom{n}{k}$ is the positive integer given by the formula

$$\binom{n}{k} := \frac{n!}{k!(n-k)!} = \frac{1}{k!}\prod_{l=0}^{k-1}(n-l). \tag{6.1}$$

Remark that the multiplicative formula on the right-hand side of (6.1) holds even if n is replaced by an arbitrary complex number z. We may thus extend the definition of binomial coefficients as follows. For each $z \in \mathbb{C}$ and integer $k \geq 0$, let

$$\binom{z}{k} := \frac{1}{k!}\prod_{l=0}^{k-1}(z-l). \tag{6.2}$$

We will now present a generalized version of Newton's celebrated binomial theorem which exploits the generalized binomial coefficients defined above.

Theorem 14 (The Generalized Binomial Expansion). *Let x, z be complex numbers. The power series of the form*

$$\sum_{k=0}^{\infty}\binom{z}{k}x^k = 1 + \binom{z}{1}x + \binom{z}{2}x^2 + \dots.$$

converges absolutely for all complex number $z \in \mathbb{C}$ if $|x| < 1$, and for any complex number z with $\Re(z) > 0$ if $|x| = 1$. Moreover,

$$\sum_{k=0}^{\infty}\binom{z}{k}x^k = (1+x)^z,$$

where the main branch of the exponential function is used.

The above expansion can be traced back to Newton's *Principia Mathematica* [58]. But it is Gauss who gave the first satisfactory proof of the convergence of the binomial series for real numbers [38]. Abel later gave a treatment that would work for general complex numbers [1]. For a detailed proof, see [64, Theorem 7.46].

Binomial Identity 15. For all complex number z and all nonnegative integer k,

$$\binom{z}{k} = (-1)^k\binom{k-z-1}{k}.$$

Proof.

$$\begin{aligned}
(-1)^k\tbinom{k-z-1}{k} &= (-1)^k\frac{1}{k!}\prod_{l=0}^{k-1}\big(k-z-1-l\big)\\
&= (-1)^k\frac{1}{k!}\prod_{l=0}^{k-1}\big(-z+(k-1)-l\big)\\
&= \frac{1}{k!}\prod_{l=0}^{k-1}\big(z-(k-1)+l\big)\\
&= \frac{1}{k!}\prod_{l=0}^{k-1}(z-l).
\end{aligned}$$

□

In order to be able to present a further generalization of the binomial coefficients, we need to digress a moment and introduce the Γ function and recall some of its key properties. The Γ function is defined via convergent improper integral

$$\Gamma(z) := \int_0^\infty x^{z-1}e^{-z}\,dx, \qquad \Re(z) > 0.$$

Using integration by parts, one can show that

$$\Gamma(z+1) = z\Gamma(z). \tag{6.3}$$

It follows from (6.3) and the fact that $\Gamma(1) = 1$ that $\Gamma(n) = (n-1)!$ for all positive integers n. The identity (6.3) can be used to uniquely extend the integral formulation for $\Gamma(z)$ to a function that is holomorphic in the whole complex plane except for zero and the negative integers, where the function has simple poles.

Theorem 16 (Euler's Reflection Formula). *For $z \in \mathbb{C}\setminus\mathbb{Z}$ the following result holds:*

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}.$$

For a proof, see [63, Theorem 3.3]. For any complex numbers z and w, the binomial coefficient $\binom{z}{w}$ can be defined as follows:

$$\binom{z}{w} := \lim_{\zeta\to z}\lim_{\eta\to w}\frac{\Gamma(\zeta+1)}{\Gamma(\eta+1)\Gamma(\zeta-\eta+1)}. \tag{6.4}$$

It is worth to emphasize that the limits in (6.4), which are rendered necessary by the singularities of the Γ function at $0, -1, -2, \dots$, cannot be interchanged.

Note that if neither z nor w is an integer, then $\binom{z}{w}$ is well-defined and we may omit both limits, whereas if $w = 0, -1, -2, \dots$, then $\binom{z}{w} = 0$ regardless of the value of z because the limit in ω is evaluated first. If w is not an integer and $z = 0, -1, -2, \dots$, then $\binom{z}{w} = \infty$ because the denominator is bounded but the numerator goes to infinity as ζ tends to z.

It remains to be shown that if w is a nonnegative integer then formula (6.4) gives the same values as formula (6.2). This is a non-trivial result for which we will provided a full proof as this will allow us to use both these formulas interchangeably. Suppose k is a nonnegative integer. We apply the reflection identity to $\Gamma(\zeta+1)$ and $\Gamma(\zeta-\omega+1)$ to get

$$\begin{aligned}
&\lim_{\zeta\to z}\lim_{\omega\to k}\frac{\Gamma(\zeta+1)}{\Gamma(\omega+1)\Gamma(\zeta-\omega+1)}\\
=\ &\lim_{\zeta\to z}\frac{\Gamma(\zeta+1)}{\Gamma(k+1)\Gamma(\zeta-k+1)}\\
=\ &\lim_{\zeta\to z}\frac{\sin\big(\pi(-\zeta+k)\big)}{\sin\big(\pi(-\zeta)\big)}\frac{\Gamma(-\zeta+k)}{\Gamma(k+1)\Gamma(-\zeta)}\\
=\ &\lim_{\zeta\to z}\frac{\sin(-\pi\zeta)\cos(k\pi)+\cos(-\pi\zeta)\sin(k\pi)}{\sin\big(-\pi\zeta\big)}\frac{\Gamma(-\zeta+k)}{\Gamma(k+1)\Gamma(-\zeta)}\\
=\ &\cos(k\pi)\frac{\Gamma(-z+k)}{k!\cdot\Gamma(-z)}\\
=\ &(-1)^k\frac{1}{k!}\prod_{l=0}^{k-1}\big(-z+k-1-l\big)\\
=\ &\frac{1}{k!}\prod_{l=0}^{k-1}(z-l),
\end{aligned}$$

where the penultimate equality follows from iterated use of (6.3) and the last equality is established as in the proof of the Binomial Identity 15. To conclude this section, we present three results that shall prove useful later on.

Binomial Identity 17. For all complex numbers z, w, we have

$$\binom{z}{w} = \binom{z}{z-w}.$$

Proof. This is a trivial consequence of formula (6.4). □

Binomial Identity 18. For all nonnegative integer k,

$$\binom{2k}{k} = (-4)^k\binom{-1/2}{k}.$$

Proof.

$$\begin{aligned}(-4)^k\binom{-1/2}{k} &= (-4)^k\frac{1}{k!}\prod_{l=0}^{k-1}(-1/2-l)\\ &= 2^k\frac{1}{k!}\prod_{l=0}^{k-1}(1+2l)\\ &= 2^k\frac{1}{k!}\frac{(2k)!}{\prod_{l=1}^{k}2l}\\ &= \frac{(2k)!}{k!k!} = \binom{2k}{k}.\end{aligned}$$

□

Theorem 19 (Vandermonde's Generalized Theorem). *For complex numbers z, w with $\Re(z+w) > -1$, $\Re(z) < -1$ and $\Re(w) > -1$, the series*

$$\sum_{l=0}^{\infty}\binom{z}{l}\binom{w}{\alpha-l}$$

converges absolutely and uniformly on compact sets of the complex plane. Moreover,

$$\sum_{l=0}^{\infty}\binom{z}{l}\binom{w}{\alpha-l} = \binom{z+w}{\alpha}, \qquad (\alpha\in\mathbb{C}).$$

For a proof, see [45, Theorem 1.2].

7. Proof of Theorem 12

In order to enhance clarity, the proof is divided into multiple steps. In Step 1, we reduce the left Riemann sum of $\sin^p(\pi x)$ to an expression involving generalized binomial coefficients. In Steps 2 to 4 we carry out some computations that will be used in Step 5, in which all these pieces are assembled.

Step 1: We begin by applying the Pythagorean trigonometric identity to $L_n\big(\sin^p(\pi x)\big)$ to get

$$\frac{1}{n}\sum_{k=0}^{n-1}\sin^p\left(\tfrac{k\pi}{n}\right) = \frac{1}{n}\sum_{k=0}^{n-1}\left(\sin^2\left(\tfrac{k\pi}{n}\right)\right)^{\frac{p}{2}} = \frac{1}{n}\sum_{k=0}^{n-1}\left(1-\cos^2\left(\tfrac{k\pi}{n}\right)\right)^{\frac{p}{2}}.$$

The summand of the term at the rightmost end being of the form $(1+x)^z$ for z satisfying $\Re(z) > 0$ and for $|x| \le 1$. Hence, Theorem 14 applies and we have

$$\frac{1}{n}\sum_{k=0}^{n-1}\left(1-\cos^2\left(\tfrac{k\pi}{n}\right)\right)^{\frac{p}{2}} = \frac{1}{n}\sum_{j=0}^{\infty}(-1)^j\binom{p/2}{j}\sum_{k=0}^{n-1}\cos^{2j}\left(\tfrac{k\pi}{n}\right).$$

But, by the Trigonometric Identity 13,

$$\frac{1}{n}\sum_{j=0}^{\infty}(-1)^j\binom{p/2}{j}\sum_{k=0}^{n-1}\cos^{2j}\left(\tfrac{k\pi}{n}\right) \;=\; \sum_{j=0}^{\infty}\left(-\tfrac{1}{4}\right)^j\binom{p/2}{j}\sum_{k=-\lfloor j/n\rfloor}^{\lfloor j/n\rfloor}\binom{2j}{j+kn}. \tag{7.1}$$

By the Symmetry Identity for binomial coefficients, we have $\binom{2j}{j+kn} = \binom{2j}{j-kn}$. Therefore,

$$\sum_{k=-\lfloor j/n\rfloor}^{\lfloor j/n\rfloor}\binom{2j}{j+kn} \;=\; \begin{cases}\binom{2j}{j}, & \text{if } 0\le j<n,\\ \binom{2j}{j}+2\sum_{k=1}^{\lfloor j/n\rfloor}\binom{2j}{j+kn}, & \text{otherwise.}\end{cases}$$

For the sake of notational convenience, we shall at times commit a deliberate slight abuse of notation and write

$$\sum_{k=-\lfloor j/n\rfloor}^{\lfloor j/n\rfloor}\binom{2j}{j+kn} \;=\; \binom{2j}{j}+2\sum_{k=1}^{\lfloor j/n\rfloor}\binom{2j}{j+kn} \tag{7.2}$$

even in situations where $0\le j<n$. When this occurs, the sum over a vacuous set of indices should be interpreted as being null.

By substituting in (7.1), we obtain

$$\begin{aligned} L_n\big(\sin^p(\pi x)\big) \;&=\; \sum_{j=0}^{\infty}\left(-\tfrac{1}{4}\right)^j\binom{p/2}{j}\sum_{k=-\lfloor j/n\rfloor}^{\lfloor j/n\rfloor}\binom{2j}{j+kn}\\ &=\; \sum_{j=0}^{\infty}\left(-\tfrac{1}{4}\right)^j\binom{p/2}{j}\left(\binom{2j}{j}+2\sum_{k=1}^{\lfloor j/n\rfloor}\binom{2j}{j+kn}\right)\\ &=\; \sum_{j=0}^{\infty}\left(-\tfrac{1}{4}\right)^j\binom{p/2}{j}\binom{2j}{j}+\sum_{j=n}^{\infty}2\left(-\tfrac{1}{4}\right)^j\binom{p/2}{j}\sum_{k=1}^{\lfloor j/n\rfloor}\binom{2j}{j+kn}. \end{aligned}$$

But, using the Binomial Identity 18 and then the Symmetry Identity 17, we find that

$$\left(-\tfrac{1}{4}\right)^j\binom{2j}{j} \;=\; \binom{-1/2}{j} \;=\; \binom{-1/2}{-1/2-j}.$$

Thus,

$$\sum_{j=0}^{\infty}\left(-\tfrac{1}{4}\right)^j\binom{p/2}{j}\binom{2j}{j} \;=\; \sum_{j=0}^{\infty}\binom{p/2}{j}\binom{-1/2}{-1/2-j} \;=\; \binom{p/2-1/2}{-1/2},$$

where the last equality comes from an application of the generalized Vandermonde identity (Theorem 19) with $z=p/2$, $w=-1/2$, and $\alpha=-1/2$. Hence, putting all the above identities together, we get

$$L_n\big(\sin^p(\pi x)\big) \;=\; \binom{p/2-1/2}{-1/2}+\sum_{j=n}^{\infty}2\left(-\tfrac{1}{4}\right)^j\binom{p/2}{j}\sum_{k=1}^{\lfloor j/n\rfloor}\binom{2j}{j+kn}. \tag{7.3}$$

Step 2: Given an integer $j \geq n \geq 1$, we set $B_j := 2\left(-\frac{1}{4}\right)^j \binom{p/2}{j}$. It follows from the Binomial Identity 15 and from formula (6.4) that

$$B_j = \frac{2}{4^j}(-1)^j \binom{p/2}{j} = \frac{2}{4^j}\binom{j-p/2-1}{j} = \frac{2}{4^j}\frac{\Gamma(j-p/2)}{j!\Gamma(-p/2)}.$$

Under the hypothesis that $p \in (0,2)$, we have $\Gamma(-p/2) < 0$. Moreover, we have $j - p/2 > 0$ for all j and it follows that $\Gamma(j-p/2) \geq 1$. This shows that $B_j \leq 0$ for all $p \in (0,2)$ and all j.

Step 3: Given an integer $j \geq n+1$, we set

$$C_j := \sum_{k=1}^{\lfloor j/(n+1) \rfloor} \left(\binom{2j}{j+k(n+1)} - \binom{2j}{j+kn} \right).$$

Observe that for every nonnegative integer k with $1 \leq k \leq \lfloor j/(n+1) \rfloor$,

$$\begin{aligned} \binom{2j}{j+k(n+1)} - \binom{2j}{j+kn} &= \frac{(2j)!}{\big(j+k(n+1)\big)!\big(j-k(n+1)\big)!} - \frac{(2j)!}{(j+kn)!(j-kn)!} \\ &= \frac{(2j)!}{(j+kn)!\big(j-k(n+1)\big)!}\left(\prod_{l=1}^{k}\frac{1}{(j+kn+l)} - \prod_{l=1}^{k}\frac{1}{(j-kn-l+1)}\right). \end{aligned}$$

One can easily verify that $\frac{(2j)!}{(j+kn)!\big(j-k(n+1)\big)!} \geq 0$. Moreover,

$$\left(\prod_{l=1}^{k}\frac{1}{(j+kn+l)} - \prod_{l=1}^{k}\frac{1}{(j-kn-l+1)}\right) \leq 0 \iff \prod_{l=1}^{k}\frac{(j-kn-l+1)}{(j+kn+l)} \leq 1,$$

and the last inequality clearly holds true. Thus, for a given $j \geq n+1$,

$$\binom{2j}{j+k(n+1)} - \binom{2j}{j+kn} \leq 0$$

for all nonnegative integer k with $1 \leq k \leq \lfloor j/(n+1) \rfloor$, and it follows that $C_j \leq 0$ for all $j \geq n+1$.

Step 4: For any integer $j \geq n+1$ and any integer k satisfying $1 \leq k \leq \lfloor j/(n+1) \rfloor$, we have that $j + kn \leq 2j$. Therefore, the values of the binomial coefficients of the form $\binom{2j}{j+kn}$ are positive. Since $\lfloor j/(n+1) \rfloor \leq \lfloor j/n \rfloor$, it follows that

$$\sum_{k=1}^{\lfloor j/(n+1) \rfloor} \binom{2j}{j+kn} \leq \sum_{k=1}^{\lfloor j/n \rfloor} \binom{2j}{j+kn}.$$

Therefore,

$$\sum_{k=1}^{\lfloor j/(n+1) \rfloor} \binom{2j}{j+k(n+1)} - \sum_{k=1}^{\lfloor j/(n+1) \rfloor} \binom{2j}{j+kn} \geq \sum_{k=1}^{\lfloor j/(n+1) \rfloor} \binom{2j}{j+k(n+1)} - \sum_{k=1}^{\lfloor j/n \rfloor} \binom{2j}{j+kn}.$$

Hence, using the fact – established in Step 2 – that $B_j \leq 0$ for all j, we obtain

$$B_j \sum_{k=1}^{\lfloor j/(n+1) \rfloor} \left(\binom{2j}{j+k(n+1)} - \binom{2j}{j+kn} \right) \leq B_j \left(\sum_{k=1}^{\lfloor j/(n+1) \rfloor} \binom{2j}{j+k(n+1)} - \sum_{k=1}^{\lfloor j/n \rfloor} \binom{2j}{j+kn} \right).$$

Step 5: Consider the difference $L_{n+1}\big(\sin^p(\pi x)\big) - L_n\big(\sin^p(\pi x)\big)$. Using the inequality obtained in Step 4, we have

$$\begin{aligned}
&L_{n+1}\big(\sin^p(\pi x)\big) - L_n\big(\sin^p(\pi x)\big)\\
&= \tbinom{p/2-1/2}{-1/2} + \sum_{j=n+1}^{\infty} 2\left(-\tfrac{1}{4}\right)^j \tbinom{p/2}{j} \sum_{k=1}^{\lfloor j/(n+1)\rfloor} \tbinom{2j}{j+k(n+1)}\\
&\qquad - \tbinom{p/2-1/2}{-1/2} + \sum_{j=n}^{\infty} 2\left(-\tfrac{1}{4}\right)^j \tbinom{p/2}{j} \sum_{k=1}^{\lfloor j/n\rfloor} \tbinom{2j}{j+kn}\\
&= \sum_{j=n}^{\infty} B_j \left(\sum_{k=1}^{\lfloor j/(n+1)\rfloor} \tbinom{2j}{j+k(n+1)} - \sum_{k=1}^{\lfloor j/n\rfloor} \tbinom{2j}{j+kn} \right)\\
&\geq \sum_{j=n+1}^{\infty} B_j \sum_{k=1}^{\lfloor j/(n+1)\rfloor} \left(\tbinom{2j}{j+k(n+1)} - \tbinom{2j}{j+kn} \right)\\
&= \sum_{j=n+1}^{\infty} B_j C_j.
\end{aligned}$$

It follows from Step 2 and Step 3 that $B_j C_j \geq 0$ for all j. Hence,

$$L_{n+1}\big(\sin^p(\pi x)\big) - L_n\big(\sin^p(\pi x)\big) \;\geq\; 0,$$

regardless of the positive integer n, and this completes the proof that the left Riemann sum of $\sin^p(\pi x)$ with respect to the uniform partition of $[0,1]$ is increasing relative to n for every $p \in (0,2)$.

8. Concluding remarks

For all $p > 2$, the left and right Riemann sums of $\sin^p(\pi x)$ are neither monotonically increasing, nor monotonically decreasing. Indeed, $L_1\big(\sin^p(\pi x)\big) = 0$, and $L_2\big(\sin^p(\pi x)\big) = \frac{1}{2}$, but

$$L_n\big(\sin^p(\pi x)\big) \xrightarrow[n\to\infty]{} \int_0^1 \sin^p(\pi x)\,dx \;<\; \int_0^1 \sin^2(\pi x)\,dx \;=\; \tfrac{1}{2}.$$

In any case, one can show that, as was proven to be the case for $p = 2$, the left and right Riemann sums of $\sin^{2m}(\pi x)$, $m \geq 1$, with respect to the uniform partition of $[0,1]$ are eventually constant.

The demonstration is based on the following trigonometric identity which can be deduced using De Moivre's formula, Euler's formula and the binomial theorem.

Trigonometric Identity 20. For all positive integer m,

$$\sin^{2m}(x) \;=\; \tfrac{1}{2^{2m}}\tbinom{2m}{m} + \tfrac{(-1)^m}{2^{2m-1}} \sum_{j=0}^{m-1} (-1)^j \tbinom{2m}{j} \cos\big(2(m-j)x\big).$$

Proposition 21. *Given any positive integer m, the left (resp. right) Riemann sums of $\sin^{2m}(\pi x)$ with respect to the uniform partition of $[0,1]$ are constant for all $n > m$ and are equal to $\frac{1}{2^{2m}}\binom{2m}{m}$.*

Proof. Given $n > m$, we have

$$\begin{aligned}
L_n\big(\sin^{2m}(\pi x)\big) &= \frac{1}{n}\sum_{k=0}^{n-1}\sin^{2m}\left(\tfrac{k\pi}{n}\right)\\
&= \frac{1}{n}\sum_{k=0}^{n-1}\left(\tfrac{1}{2^{2m}}\tbinom{2m}{m} + \tfrac{(-1)^m}{2^{2m-1}}\sum_{j=0}^{m-1}(-1)^j\tbinom{2m}{j}\cos\left(\tfrac{2(m-j)k\pi}{n}\right)\right)\\
&= \tfrac{1}{2^{2m}}\tbinom{2m}{m} + \frac{1}{n}\sum_{k=0}^{n-1}\left(\tfrac{(-1)^m}{2^{2m-1}}\sum_{j=0}^{m-1}(-1)^j\tbinom{2m}{j}\cos\left(\tfrac{2(m-j)k\pi}{n}\right)\right)\\
&= \tfrac{1}{2^{2m}}\tbinom{2m}{m} + \frac{1}{n}\frac{(-1)^m}{2^{2m-1}}\sum_{j=0}^{m-1}(-1)^j\tbinom{2m}{j}\sum_{k=0}^{n-1}\cos\left(\tfrac{2(m-j)k\pi}{n}\right)\\
&= \tfrac{1}{2^{2m}}\tbinom{2m}{m} + \frac{1}{n}\frac{(-1)^m}{2^{2m-1}}\sum_{j=0}^{m-1}(-1)^j\tbinom{2m}{j}\Re\left(\sum_{k=0}^{n-1}\exp\left(\tfrac{i2(m-j)k\pi}{n}\right)\right)\\
&= \tfrac{1}{2^{2m}}\tbinom{2m}{m} + \frac{1}{n}\frac{(-1)^m}{2^{2m-1}}\sum_{j=0}^{m-1}(-1)^j\tbinom{2m}{j}\Re\left(\frac{1-\exp\big(i2(m-j)\pi\big)}{1-\exp\left(\frac{i2(m-j)\pi}{n}\right)}\right)\\
&= \tfrac{1}{2^{2m}}\tbinom{2m}{m},
\end{aligned}$$

as, under the hypothesis on n, we have $\exp\big(i2\pi(m-j)\big) = 1$ but $\exp\left(\frac{i2\pi(m-j)}{n}\right) \neq 1$. □

Upon analyzing the proof of Theorem 12 given above, we see that the actual value of p only plays a significant role in Step 2 when we show that

$$B_j = \frac{2}{4^j}\frac{\Gamma(j-p/2)}{j!\Gamma(-p/2)}.$$

Remark that $\Gamma(-p/2) < 0$ if $p \in \bigcup_{m=0}^{\infty}(4m, 4m+2)$, and $\Gamma(-p/2) > 0$ if $p \in \bigcup_{m=0}^{\infty}(4m+2, 4m+4)$. As for $\Gamma(j-p/2)$, it is positive for all integer $j > p/2$. Hence, if p belongs to $\bigcup_{m=0}^{\infty}(4m, 4m+2)$, then the proof of Theorem 12 applies and shows that the sequence $x_n := L_n\big(\sin^p(\pi x)\big)$ is monotonically increasing for $n > \lceil \frac{p}{2} \rceil$. If, instead, p belongs to $\bigcup_{m=0}^{\infty}(4m+2, 4m+4)$, then, by performing slight modifications to proof of Theorem 12 to account for the fact that the B_j's are now positive, one can show that the sequence $x_n := L_n\big(\sin^p(\pi x)\big)$ is monotonically decreasing for $n > \lceil \frac{p}{2} \rceil$.

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