Convolution weights on $\ell_2$-spaces

Frédéric Morneau-Guérin
Frederic.Morneau-Guerin@teluq.ca

TRENDS IN OPERATOR THEORY AND ITS APPLICATIONS 2021
Instituto Superior Técnico, Universidade de Lisboa
Lisboa, Portugal

June 25 2021
This is joint work with Thomas J. Ransford and Colin Krawchuk.
$G$: Unimodular locally compact group.
• $G$: Unimodular locally compact group.

• Weight function: Function $w: G \rightarrow (0, \infty)$. 
• $G$: Unimodular locally compact group.

• Weight function: Function $w: G \to (0, \infty)$.

• Weighted $L_p$-space on $G$ with weight $w$:

\[ L_p(G, w) := \left\{ f: G \to \mathbb{C} : \|f\|_{p,w} < \infty \right\}. \]
- \( G \) : Unimodular locally compact group.

- **Weight function** : Function \( w : G \to (0, \infty) \).

- **Weighted \( L_p \)-space on \( G \) with weight \( w \) :**

\[
L_p(G, w) := \left\{ f : G \to \mathbb{C} : \| f \|_{p,w} < \infty \right\}.
\]

- \( \| f \|_{p,w} := \| w f \|_p \).
Let $f, g : G \to \mathbb{C}$ be two functions. Their convolution $f \ast g : G \to \mathbb{C}$ is the “function” defined by the formula

$$(f \ast g)(x) := \int_G f(y)g(y^{-1}x) \, dL\lambda(y), \quad (x \in G).$$

**Question 1**

For $p \geq 1$, what can we say about the class $\mathcal{W} \mathcal{L}_p$ of groups $G$ for which there exists a weight function $w$ such that $L_p(G, w)$ is $\ast$-stable?

**Question 2**

Given $p \geq 1$ and $G \in \mathcal{W} \mathcal{L}_p$, what can we say about the class $\mathcal{W}_p(G)$ of weight functions on $G$ for which $L_p(G, w)$ is $\ast$-stable?
Partial answer to question 2

1. \[ \| f \ast g \|_{1,w} \leq C \| f \|_{1,w} \| g \|_{1,w}, \quad \forall f, g \in L_1(G, w) \]
   if and only if \( w : G \to \mathbb{C} \) is weakly-submultiplicative, i.e.
   \[ w(xy) \leq Cw(x)w(y), \quad \text{a.e.} x, y \in G. \]

2. \[ \| f \ast g \|_{p,w} \leq C \| f \|_{p,w} \| g \|_{p,w}, \quad \forall f, g \in L_p(G, w) \]
   if \( w^{-q} \) is weakly-subconvolutive, i.e.
   \[ (w^{-q} \ast w^{-q})(x) \leq C^q w^{-q}(x), \quad \text{a.e.} x \in G. \]
Definition & notation.

1. The *stability index* of \((G, w)\) :

\[ C(G, w) := \sup \left\{ \frac{\|f \ast g\|_w}{\|f\|_w \|g\|_w} : f, g \in \ell_2(G, w) \right\}. \]

2. The *subconvolutivity index* of \((G, w)\) :

\[ C_2(G, w)^2 := \sup_{x \in G} \frac{(w^{-2} \ast w^{-2})(x)}{w^{-2}(x)}. \]

Theorem. \( C(G, w) \leq C_2(G, w) \).
Main problem.
Does there exist a discrete abelian group $G$ and a weight function $w$ on $G$ such that

- $\ell_2(G, w)$ is a convolution algebra.
- $w^{-2}$ is not weakly subconvolutive, i.e.

$$C_2(G, w)^2 = \sup_{x \in G} \frac{(w^{-2} * w^{-2})(x)}{w^{-2}(x)} = \sup_{x \in G} \sum_{y \in G} \frac{w^2(x)}{w^2(y)w^2(y^{-1}x)} = \infty.$$
Theorem 1. (Kuznetsova)
Suppose that $\ell_2(G, w)$ is $\ast$-stable. Then $\sum_{x \in G} \frac{1}{w(x)w(x^{-1})} < \infty$.

Sketch of proof.

- $\ell_2(G, w)$ admits a character $\chi : \ell_2(G, w) \to \mathbb{C}$.
- Riesz’s repr. thm: $\chi(f) = \langle f, h \rangle_w = \sum_{x \in G} w(x)^2 f(x)h(x)$.
- Taking $f := \delta_x$ we find that $h(x) = \frac{\chi(\delta_x)}{w(x)^2}$.
- $\sum_{x \in G} \frac{\chi(\delta_x)^2}{w(x)^2} = \sum_{x \in G} w(x)^2 |h(x)|^2 = \|h\|_w^2 < \infty$.
- We may write $1 = \chi(\delta_0) = \chi(\delta_x \ast \delta_{x^{-1}}) = \chi(\delta_x)\chi(\delta_{x^{-1}})$.

Hence by C-S,

$$\sum_{x \in G} \frac{1}{w(x)w(x^{-1})} \leq \left( \sum_{x \in G} \frac{\chi(\delta_x)^2}{w(x)^2} \right)^{1/2} \left( \sum_{x \in G} \frac{\chi(\delta_{x^{-1}})^2}{w(x^{-1})^2} \right)^{1/2} < \infty.$$
Under certain regularity hypotheses on $w$, the necessary condition coincide with the sufficient one.

**Theorem 2.** (Ransford–M-G) 
Assume that there exists a constant $M > 0$ such that $\forall x, y \in G$

$$w(xy) \leq M(w(x) + w(y)) \quad \& \quad w(x^{-1}) \leq Mw(x).$$

Then the following statements are equivalent:

1. $C_2(G, w) < \infty$;
2. $C(G, w) < \infty$;
3. $\sum_{x \in G} \frac{1}{w(x)w(x^{-1})} < \infty$;
4. $\sum_{x \in G} \frac{1}{w(x)^2} < \infty$. 

A new approach.

- Characterize $C(G, w)$ as the norm of a certain operator.
- Exploit this characterization to obtain concrete estimates of $C(G, w)$. 
Notation.

- $H := \ell_2(G)$.
- $H_0 := \text{span}\{e_x : x \in G\}$, where
  \[
e_x(y) := \begin{cases} 
  1, & y = x; \\
  0, & y \neq x.
  \end{cases}
\]
- $\text{Lin}(H_0) := \{T : H_0 \to H_0 \mid T \text{ linear}\}$.
- $\mathcal{B}(H_0) := \{T : H_0 \to H_0 \mid T \text{ linear and } \| \cdot \|_H\text{-bounded}\}$.
- $e_y \otimes e_z \in \mathcal{B}(H_0)$ is the rank-1 operator defined by
  \[
  (e_y \otimes e_z)(h) := \langle h, e_z \rangle e_y = h(z)e_y, \quad (h \in H).
  \]
Consider the map \( T_w : H_0 \to \text{Lin}(H_0) \) defined by

\[
T_w(h) := \sum_{y \in G} \sum_{z \in G} h(y + z) \frac{w(y + z)}{w(y)w(z)} (e_y \otimes e_z).
\]

**Theorem 3.** (Ransford–M-G)
The following statements are equivalent:

1. \( \ell_2(G, w) \) is \(*\)-stable;
2. \( T_w \) is a bounded linear map from \( H_0 \) into \( \mathcal{B}(H_0) \).

In this case

\[
C(G, w) = \| T_w : H_0 \to \mathcal{B}(H_0) \|.
\]
If $T_w$ is a bounded linear map of $H_0$ into $\mathcal{B}(H_0)$, then there is a (unique) bounded linear map $\tilde{T}_w : H \to \mathcal{B}(H)$ that extends $T_w$ in the sense that

$$\tilde{T}_w(h) \upharpoonright H_0 = T_w(h), \quad (h \in H_0).$$

Moreover

$$\|\tilde{T}_w : H \to \mathcal{B}(H)\| = \|T_w : H_0 \to \mathcal{B}(H_0)\|.$$

**Theorem 4.** (Ransford–M-G)
The following statements are equivalent:

1. $C_2(G, w) < \infty$.
2. $\tilde{T}_w$ is a bounded linear map from $H$ into $\mathcal{S}_2(H)$, the class of Hilbert–Schmidt operators on $H$;

In this case

$$C_2(G, w) = \|\tilde{T}_w : H \to \mathcal{S}_2(H)\|.$$
Computing $\|\tilde{T}_w : H \to \mathcal{B}(H)\|$ can be quite complicated.

We seek an estimate for $C(G, w)$ which is smaller than $\|\tilde{T}_w : H \to \mathcal{S}_2(H)\|$, yet simpler to compute than $\|\tilde{T}_w : H \to \mathcal{B}(H)\|$.

One possibility is to consider $\|\tilde{T}_w : H \to \mathcal{S}_p(H)\|$ for the Schatten classes $\mathcal{S}_p(H)$ with $p \in (2, \infty)$. 
**Definition.**
Given a weighted group \((G, w)\) and \(p \in (2, \infty)\), set

\[
C_p(G, w) := \|\widetilde{T}_w : H \to \mathcal{S}_p(H)\|.
\]

**Theorem 5.** (Ransford–M-G)

\[
C_4(G, w)^8 \leq \sup_{s \in G} \sum_{x \in G} \sum_{y \in G} \left( \sum_{z \in G} \frac{w(s)w(x)w(y)w(y^{-1}xs)}{w(z^{-1}x)^2w(z^{-1}y)^2w(z)^2w(y^{-1}z_s)^2} \right)^2.
\]
Theorem 6. (Ransford–M-G)
For $n \geq 3$, define $w : \mathbb{Z}_n \to (0, \infty)$ by

$$w(0) = w(1) = \ldots = w(n-2) := 1 \quad \text{and} \quad w(n-1) := t \geq 1.$$ 

We have

$$\frac{C(\mathbb{Z}_n, w)}{C_2(\mathbb{Z}_n, w)} \leq \left( \frac{(n-1)+t^2}{2+(n-2)t^2} \right)^{1/2}.$$
Figure – Graph of $C(\mathbb{Z}_n, w)/C_2(\mathbb{Z}_n, w)$ for various values of $n$. 
Idea.
Given $\{(G_n, w_n)\}_{n=1}^{\infty}$ a countable family of discrete abelian groups with weights, can we define a weight function $w$ on $G := \bigoplus_{n \geq 1} G_n$ in such a way that the value of $C(G, w)$ can be inferred from that of $C(G_n, w_n)$?
First attempt.

Consider the weight function $w$ on $G := \bigoplus_{n \geq 1} G_n$ defined by

$$w(x) \sim \prod_{n=1}^{\infty} w_n(x_n).$$
Theorem 7. (Ransford–M-G)

Given \( \{(G_n, w_n)\}_{n=1}^{\infty} \) a family of discrete abelian groups of finite order with weights satisfying \( w_n(0_n) = 1 \) for all \( n \geq 1 \).

If \( G := \bigoplus_{n \geq 1} G_n \) and \( w(x) := \prod_{n=1}^{\infty} w_n(x_n) \), then

\[
C(G, w) \geq \prod_{i=1}^{k} C(G_i, w_i), \quad (\forall k \geq 1).
\]
Theorem 7. (Ransford–M-G, 2019)
Given \( \{(G_n, w_n)\}_{n=1}^{\infty} \) a family of discrete abelian groups of finite order with weights satisfying \( w_n(0_n) = 1 \) for all \( n \geq 1 \).

If \( G := \bigoplus_{n \geq 1} G_n \) and \( w(x) := \prod_{n=1}^{\infty} w_n(x_n) \), then

\[
C(G, w) \geq \prod_{i=1}^{k} C(G_i, w_i), \quad (\forall k \geq 1).
\]

Lemma 8. For all \( n \geq 1 \), we have

\[
C(G_n, w_n) \geq \frac{2}{\sqrt{3}}.
\]
Second attempt.

- Consider the weight function $w$ on $G := \bigoplus_{n \geq 1} G_n$ defined by

$$w(x) \sim \sup_{n \geq 1} w_n(x_n).$$
Example 1.
We define $w_n : \mathbb{Z}_3 \to (0, \infty)$ inductively as follows:

- For $n = 1$,
  - $0 \mapsto 1$,
  - $1 \mapsto 4$,
  - $2 \mapsto 16$.

- For $n \geq 2$,
  - $0 \mapsto 1$,
  - $1 \mapsto 2(3^{n-1})^{1/2} \cdot w_{n-1}(2)$,
  - $2 \mapsto w_n(1)^2$.

We then endow $G := \bigoplus_{n \geq 1} G_n$ with the following weight function:

$$\{x_n\}_{n=1}^{\infty} \mapsto \sup_{n \geq 1} w_n(x_n).$$
Theorem 9. (Kuznetsova–Krawchuk–M-G)
Given \( \{(G_n, w_n)\}_{n=1}^{\infty} \) a family of discrete abelian groups of finite order with weights and given \( G := \bigoplus_{n \geq 1} G_n \) endowed with \( \{x_n\}_{n=1}^{\infty} \mapsto \sup_{n \geq 1} w_n(x_n) \). If

- \( w_n(0_n) = 1 \).
- For all \( n \geq 1 \), we have \( w_n(xy) \leq w_n(x)w_n(y) \) for all \( x, y \in G_n \).
- For all \( n > 1 \), there exists \( a_n, b_n \in G_n \) such that

\[
    w_n(a_nb_n) = w_n(a_n)w_n(b_n) \quad \& \quad w_n(a_n), w_n(b_n) \geq \max_{z \in G_{n-1}} w_{n-1}(z).
\]

Then \( C(G, w) = \infty \).
Example 2.
We define $w_n : \mathbb{Z}_3 \rightarrow (0, \infty)$ inductively as follows:

- For $n = 1$,
  
  $0 \mapsto 1$, \\
  $1 \mapsto 2$, \\
  $2 \mapsto 4$.

- For $n \geq 2$,
  
  $0 \mapsto 1$, \\
  $1 \mapsto 2(3^{n-1})^{1/2} \cdot w_{n-1}(2)$, \\
  $2 \mapsto \left(\frac{n}{3^{n-1}}\right)^{1/2} \cdot w_n(1)^2$.

We then endow $G := \bigoplus_{n \geq 1} G_n$ with the following weight function:

\[ \{x_n\}_{n=1}^{\infty} \mapsto \sup_{n \geq 1} w_n(x_n) \]
References.