Convolution weights on ℓ_2-spaces

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- $\|f\|_{p,w} := \ \|wf\|_p$.

Let $f, g : G \to \mathbb{C}$ be two functions. Their *convolution* $f * g : G \to \mathbb{C}$ is the "function" defined by the formula

$$(f * g)(x) := \int_G f(y) g(y^{-1}x) \, d_L\lambda(y), \qquad (x \in G).$$

Question 1
For $p \geq 1$, what can we say about the class $\mathscr{W}\mathscr{L}_p$ of groups G for which there exists a weight function w such that $L_p(G, w)$ is $*$-stable ?

Question 2
Given $p \geq 1$ and $G \in \mathscr{W}\mathscr{L}_p$, what can we say about the class $\mathscr{W}_p(G)$ of weight functions on G for which $L_p(G, w)$ is $*$-stable ?

Partial answer to question 2

1. $\|f * g\|_{1,w} \leq C\|f\|_{1,w}\|g\|_{1,w}, \qquad \forall f, g \in L_1(G,w)$
   if and only if $w : G \to \mathbb{C}$ is weakly-submultiplicative, i.e.

$$w(xy) \leq Cw(x)w(y), \qquad \text{a.e.} x, y \in G.$$

2. $\|f * g\|_{p,w} \leq C\|f\|_{p,w}\|g\|_{p,w}, \qquad \forall f, g \in L_p(G,w)$
   if w^{-q} is weakly-subconvolutive, i.e.

$$(w^{-q} * w^{-q})(x) \ \leq \ C^q w^{-q}(x), \qquad \text{a.e.} x \in G.$$

Definition & notation.

1. The *stability index of* (G,w) :

$$C(G,w) := \sup\left\{ \frac{\|f*g\|_w}{\|f\|_w\|g\|_w} \,:\, f,g \in \ell_2(G,w) \right\}.$$

2. The *subconvolutivity index of* (G,w) :

$$C_2(G,w)^2 := \sup_{x\in G} \frac{(w^{-2}*w^{-2})(x)}{w^{-2}(x)}.$$

Theorem. $C(G,w) \leq C_2(G,w)$.

Main problem.

Does there exist a discrete abelian group G and a weight function w on G such that

- $\ell_2(G,w)$ is a convolution algebra.
- w^{-2} is not weakly subconvolutive, i.e.

$$C_2(G,w)^2 = \sup_{x\in G} \frac{(w^{-2} * w^{-2})(x)}{w^{-2}(x)} = \sup_{x\in G} \sum_{y\in G} \frac{w^2(x)}{w^2(y)w^2(y^{-1}x)} = \infty.$$

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Theorem 1. (Kuznetsova)
Suppose that $\ell_2(G,w)$ is $*$-stable. Then $\sum_{x\in G} \frac{1}{w(x)w(x^{-1})} < \infty$.

Sketch of proof.

- $\ell_2(G,w)$ admits a character $\chi : \ell_2(G,w) \to \mathbb{C}$.
- Riesz's repr. thm : $\chi(f) = \langle f,h\rangle_w = \sum_{x\in G} w(x)^2 f(x)\overline{h(x)}$.
- Taking $f := \delta_x$ we find that $h(x) = \frac{\overline{\chi(\delta_x)}}{w(x)^2}$.
- $\sum_{x\in G} \frac{|\chi(\delta_x)|^2}{w(x)^2} = \sum_{x\in G} w(x)^2|h(x)|^2 = \|h\|_w^2 < \infty$.
- We may write $1 = \chi(\delta_0) = \chi(\delta_x * \delta_{x^{-1}}) = \chi(\delta_x)\chi(\delta_{x^{-1}})$.
  Hence by C-S,

$$\sum_{x\in G} \frac{1}{w(x)w(x^{-1})} \leq \left(\sum_{x\in G} \frac{|\chi(\delta_x)|^2}{w(x)^2}\right)^{1/2} \left(\sum_{x\in G} \frac{|\chi(\delta_{x^{-1}})|^2}{w(x^{-1})^2}\right)^{1/2} < \infty.$$

Under certain regularity hypotheses on w, the necessary condition coincide with the sufficient one.

Theorem 2. (Ransford–M-G)
Assume that there exists a constant $M > 0$ such that $\forall x, y \in G$

$$w(xy) \leq M\big(w(x) + w(y)\big) \quad \& \quad w(x^{-1}) \leq Mw(x).$$

Then the following statements are equivalent :

1. $C_2(G, w) < \infty$;
2. $C(G, w) < \infty$;
3. $\sum_{x \in G} \frac{1}{w(x)w(x^{-1})} < \infty$;
4. $\sum_{x \in G} \frac{1}{w(x)^2} < \infty$.

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A new approach.

- Characterize $C(G,w)$ as the norm of a certain operator.
- Exploit this characterization to obtain concrete estimates of $C(G,w)$.

Notation.

- $H := \ell_2(G)$.
- $H_0 := \text{span}\{e_x : x \in G\}$, where

$$e_x(y) := \begin{cases} 1, & y = x; \\ 0, & y \neq x. \end{cases}$$

- $\text{Lin}(H_0) := \{T : H_0 \to H_0 \mid T \text{ linear}\}$.
- $\mathscr{B}(H_0) := \{T : H_0 \to H_0 \mid T \text{ linear and } \|\cdot\|_H\text{-bounded}\}$.
- $e_y \otimes e_z \in \mathscr{B}(H_0)$ is the rank-1 operator defined by

$$(e_y \otimes e_z)(h) := \langle h, e_z \rangle e_y = h(z) e_y, \qquad (h \in H).$$

Consider the map $T_w : H_0 \to \mathrm{Lin}(H_0)$ defined by

$$T_w(h) := \sum_{y\in G}\sum_{z\in G} h(y+z)\frac{w(y+z)}{w(y)w(z)}(e_y \otimes e_z).$$

Theorem 3. (Ransford–M-G)

The following statements are equivalent :

1. $\ell_2(G,w)$ is $*$-stable ;
2. T_w is a bounded linear map from H_0 into $\mathscr{B}(H_0)$.

In this case

$$C(G,w) \;=\; \big\|T_w : H_0 \to \mathscr{B}(H_0)\big\|.$$

If T_w is a bounded linear map of H_0 into $\mathscr{B}(H_0)$, then there is a (unique) bounded linear map $\widetilde{T}_w : H \to \mathscr{B}(H)$ that extends T_w in the sense that

$$\widetilde{T}_w(h) \upharpoonright H_0 \;=\; T_w(h), \qquad\qquad (h \in H_0).$$

Moreover

$$\left\|\widetilde{T}_w : H \to \mathscr{B}(H)\right\| \;=\; \left\|T_w : H_0 \to \mathscr{B}(H_0)\right\|.$$

Theorem 4. (Ransford–M-G)
The following statements are equivalent :

1. $C_2(G,w) < \infty$.
2. $\widetilde{T}_w$ is a bounded linear map from H into $\mathscr{S}_2(H)$, the class of Hilbert–Schmidt operators on H ;

In this case

$$C_2(G,w) \;=\; \left\|\widetilde{T}_w : H \to \mathscr{S}_2(H)\right\|.$$

Computing $\left\|\widetilde{T}_w : H \to \mathscr{B}(H)\right\|$ can be quite complicated.

We seek an estimate for $C(G,w)$ which is smaller than $\left\|\widetilde{T}_w : H \to \mathscr{S}_2(H)\right\|$, yet simpler to compute than $\left\|\widetilde{T}_w : H \to \mathscr{B}(H)\right\|$.

One possibility is to consider $\left\|\widetilde{T}_w : H \to \mathscr{S}_p(H)\right\|$ for the Schatten classes $\mathscr{S}_p(H)$ with $p \in (2,\infty)$.

Definition.
Given a weighted group (G,w) and $p \in (2,\infty)$, set

$$C_p(G,w) := \left\|\widetilde{T}_w : H \to \mathscr{S}_p(H)\right\|.$$

Theorem 5.(Ransford–M-G)

$$C_4(G,w)^8 \leq \sup_{s\in G}\sum_{x\in G}\sum_{y\in G}\left(\sum_{z\in G}\frac{w(s)w(x)w(y)w(y^{-1}xs)}{w(z^{-1}x)^2w(z^{-1}y)^2w(z)^2w(y^{-1}zs)^2}\right)^2.$$

Theorem 6. (Ransford–M-G)
For $n \geq 3$, define $w : \mathbb{Z}_n \to (0,\infty)$ by

$$w(0) = w(1) = ... = w(n-2) := 1 \quad \text{and} \quad w(n-1) := t \;\geq 1.$$

We have

$$\frac{C(\mathbb{Z}_n, w)}{C_2(\mathbb{Z}_n, w)} \;\leq\; \left(\frac{(n-1)+t^2}{2+(n-2)t^2}\right)^{1/2}.$$

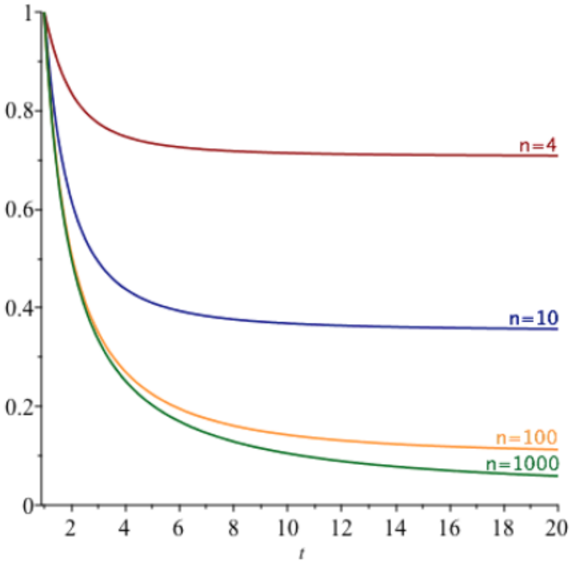

FIGURE – Graph of $C(\mathbb{Z}_n, w)/C_2(\mathbb{Z}_n, w)$ for various values of n.

Idea.

Given $\{(G_n, w_n)\}_{n=1}^{\infty}$ a countable family of discrete abelian groups with weights, can we define a weight function w on $G := \bigoplus_{n\geq 1} G_n$ in such a way that the value of $C(G,w)$ can be infered from that of $C(G_n, w_n)$?

First attempt.

- Consider the weight function w on $G := \bigoplus_{n\geq 1} G_n$ defined by

$$w(x) \ \sim \ \prod_{n=1}^{\infty} w_n(x_n).$$

Theorem 7. (Ransford–M-G)
Given $\{(G_n, w_n)\}_{n=1}^{\infty}$ a family of discrete abelian groups of finite order with weights satisfying $w_n(0_n) = 1$ for all $n \geq 1$.

If $G := \bigoplus_{n\geq 1} G_n$ and $w(x) := \prod_{n=1}^{\infty} w_n(x_n)$, then

$$C(G, w) \geq \prod_{i=1}^{k} C(G_i, w_i), \qquad (\forall k \geq 1).$$

Theorem 7. (Ransford–M-G, 2019)
Given $\{(G_n, w_n)\}_{n=1}^{\infty}$ a family of discrete abelian groups of finite order with weights satisfying $w_n(0_n) = 1$ for all $n \geq 1$.
If $G := \bigoplus_{n\geq 1} G_n$ and $w(x) := \prod_{n=1}^{\infty} w_n(x_n)$, then

$$C(G,w) \geq \prod_{i=1}^{k} C(G_i, w_i), \qquad (\forall k \geq 1).$$

Lemma 8. For all $n \geq 1$, we have

$$C(G_n, w_n) \ \geq \ \frac{2}{\sqrt{3}}.$$

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Second attempt.

- Consider the weight function w on $G := \bigoplus_{n\geq 1} G_n$ defined by

$$w(x) \ \sim \ \sup_{n\geq 1} w_n(x_n).$$

Example 1.

We define $w_n : \mathbb{Z}_3 \to (0,\infty)$ inductively as follows :

- For $n = 1$,

$$0 \mapsto 1,$$
$$1 \mapsto 4,$$
$$2 \mapsto 16.$$

- For $n \geq 2$,

$$0 \mapsto 1,$$
$$1 \mapsto 2(3^{n-1})^{1/2} \cdot w_{n-1}(2),$$
$$2 \mapsto w_n(1)^2.$$

We then endow $G := \bigoplus_{n\geq 1} G_n$ with the following weight function $\{x_n\}_{n=1}^{\infty} \mapsto \sup\limits_{n\geq 1} w_n(x_n)$.

Theorem 9. (Kuznetsova–Krawchuk–M-G)
Given $\{(G_n, w_n)\}_{n=1}^{\infty}$ a family of discrete abelian groups of finite order with weights and given $G := \bigoplus_{n\geq 1} G_n$ endowed with $\{x_n\}_{n=1}^{\infty} \mapsto \sup_{n\geq 1} w_n(x_n)$. If

- $w_n(0_n) = 1$.
- For all $n \geq 1$, we have $w_n(xy) \leq w_n(x)w_n(y)$ for all $x, y \in G_n$.
- For all $n > 1$, there exists $a_n, b_n \in G_n$ such that

$$w_n(a_n b_n) = w_n(a_n)w_n(b_n) \qquad \& \qquad w_n(a_n), w_n(b_n) \geq \max_{z\in G_{n-1}} w_{n-1}(z).$$

Then $C(G, w) = \infty$.

Example 2.

We define $w_n : \mathbb{Z}_3 \to (0,\infty)$ inductively as follows :

- For $n = 1$,

$$0 \mapsto 1, \quad 1 \mapsto 2, \quad 2 \mapsto 4.$$

- For $n \geq 2$,

$$0 \mapsto 1,$$
$$1 \mapsto 2(3^{n-1})^{1/2} \cdot w_{n-1}(2),$$
$$2 \mapsto \left(\frac{n}{3^{n-1}}\right)^{1/2} \cdot w_n(1)^2.$$

We then endow $G := \bigoplus_{n\geq 1} G_n$ with the following weight function $\{x_n\}_{n=1}^{\infty} \mapsto \sup_{n\geq 1} w_n(x_n)$

THANK YOU
for your
ATTENTION!

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