

Monotonicity Analysis over Chains and Curves

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Abstract. Chains are vector-valued signals sampling a curve. They are important to motion signal processing and to many scientific applications including location sensors. We propose a novel measure of smoothness for chains curves by generalizing the scalar-valued concept of monotonicity. Monotonicity can be defined by the connectedness of the inverse image of balls. This definition is coordinate-invariant and can be computed efficiently over chains. Monotone curves may be discontinuous, but continuous monotone curves are differentiable a.e. Over chains, a simple sphere-preserving filter is shown to never decrease the degree of monotonicity. It outperforms moving average filters over a synthetic data set. Applications include Time Series Segmentation, chain reconstruction from unordered data points, Optical Character Recognition, and Pattern Matching.

§1. Introduction

Monotonicity is one of the simplest property a signal may have. It offers a powerful qualitative description (“it goes up,” “it goes down”). Given data coming in from either sensors or from a numerical simulation, monotonicity is independent of the sampling frequency and is robust with respect to missing data [8]. Many geometrical objects such as curves are typically defined in a parametrization-independent way which makes monotonicity appealing.

In this paper, we are concerned with discretely sampled curves (which we call chains) such as the trajectory of a particle in some vector space. This problem has applications in motion capture and tracking [14, 1].

We expect a “smooth” scalar-valued signal not to change too quickly: it should be locally constant. Therefore, classical low pass filters such as the moving average (MA) are often sufficient to help smooth signals. Unfortunately, “smooth” chains are not locally constant: consider a loosely

sampled circle (see Fig. 1). Moreover, a chain may lie on a sphere or other higher dimensional surface and we may need to preserve this embedding. In Fig. 1, a chain on a circle is filtered using a moving average: we see that the filtered chain can, at best, follow a circle of a smaller radius. A filter is sphere-preserving (resp. circle-preserving) if, when the input data points are on a sphere (resp. circle), the filtered data points also lie on the same sphere (resp. circle). It is readily shown that no linear filter except the identity can be sphere-preserving (SP) or circle-preserving (CP). In general, an SP filter is CP. We offer a simple SP filter in Section 5.

One of the main contribution of this paper is to provide a generalization of the concept of monotonicity which applies to vector-valued signals and to curves. This definition is shown to be robust with respect to removal of data points and to be efficiently computed. Over curves, we show that monotone curves have many of the same properties as monotone functions as far as continuity and differentiability are concerned. We also propose a SP filter which we show to never decrease the degree of monotonicity. Experimentally, we show that the degree of monotonicity is inversely correlated with noise and we compare the SP filter with simple MA filters, proving the nonlinear SP filter is a good choice when noise levels are low. Applications of this work include chain reconstruction from unordered data points [3] and Optical Character Recognition [15].

§2. Related Work

A motion signal is comprised of two components: orientation and translation. The orientation vector indicates where the object is facing, whereas the translation component determines the object’s location. Recent work has focused on smoothing the orientation vectors [14, 12], whereas the results of the present paper apply equally well to orientation vectors (points on the surface of a unit sphere) as to arbitrary translation signals.

In [2, 7, 10], the authors chose to define monotonicity for curves or chains with an arbitrary direction vector: a curve is monotone if its projection on a line is does not backtrack. While this is a sensible choice given the lack of definition elsewhere, we argue that not all applications support an arbitrary direction that can be used to define monotonicity.

The definition of monotonicity has been extended to real-valued functions [4, 5, 6, 11, 16, 17] ($f : \mathbb{R}^n \rightarrow \mathbb{R}$) by using contour lines (or surfaces)

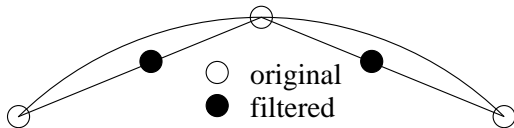


Fig. 1. Given samples on circle, a simple moving average does not preserve the embedding.

but the idea does not immediately generalize to curves and chains.

One approach to chain smoothing is to use B-splines and Bezier curves with the L^2 norm [9]. Correspondingly, we could measure the “smoothness” of a chain by measuring how closely one can fit it to a smooth curve. Our approach differs in that we do not use polygonal approximations or curve fitting: we consider chains to be first-class citizens.

§3. Monotone Curves

Recall that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to be monotone increasing if $f(x) \geq f(y)$ whenever $x \geq y$ and monotone decreasing if $f(x) \leq f(y)$ whenever $x \geq y$. A monotone increasing or monotone decreasing function is said to be monotone. Recall that $B = \{x : |x - a| \leq R\}$ is called a (closed) ball of radius R centered around a : in the multidimensional case, the ball is a generalization of the (closed) interval.

Proposition 1. $f : \mathbb{R} \rightarrow \mathbb{R}$ is monotone if and only if $f^{-1}(B)$ is connected for all balls B .

An arc-length parametrized curve $s : t \rightarrow s(t)$ is R -monotone for $R > 0$ if the inverse image of any closed ball of radius at most R , under s , is connected. Straight lines are R -monotone for all $R > 0$. As motivation the discrete case, we want to compare monotone curves with monotone functions. Monotone functions $f : \mathbb{R} \rightarrow \mathbb{R}$ are differentiable almost everywhere, and they do not have to be continuous. R -monotone also do not have to be continuous: the curve $s(t) = (f(t), f(t))$ where $f'(t) = 1$ a.e. is R -monotone for all $R > 0$. Moreover, they are also differentiable a.e. as the next proposition shows.

Proposition 2. Continuous R -monotone curves are differentiable a.e.

Proof: Take any point x in the (open) domain of the curve s . Choose another point y so that the arc-length $y - x$ over s is smaller than R . Consider any point z on s between y and x , then z must be contained in all balls of radius R containing both x and y . It follows that s must be differentiable from the left at x . Similarly, s is differentiable from the right at x . If the two derivative from the left and from the right do not match, then it is possible to find y and y' close to x from the left and the right such that there is a ball of radius R containing both y and y' but not x , a contradiction. \square

Just like monotone functions, continuous R -monotone curves do not have to be twice differentiable, consider the arc-length parametrized version of $s(t) = (t, |t|)$ for $t \in (-1, 1)$.

Differentiable functions are not necessarily monotone. Likewise differentiable curves are not necessarily R -monotone as the next proposition shows.

Proposition 3. *There is a differentiable continuous finite curves with no cross-over (that is, $t \rightarrow s(t)$ is one-to-one) which is not R -monotone for any $R > 0$.*

Proof: Consider a curve following an inward spiral around a fixed point such as $s(t) = (2\pi - t)(\cos t, \sin t)$ for $t \in (0, 2\pi]$. \square

Functions are monotone or not, and there is no “degree of monotonicity.” Similarly, for curves of finite length, it simply matters whether they are R -monotone for some finite R since R -monotonicity is scale-dependent.

Proposition 4. *Given a R -monotone curve, scaling the curve by a factor $\infty > \alpha > 0$ makes it αR -monotone.*

§4. Signal Monotonicity

In this section, we define monotonicity for vector-valued signals or chains as a natural extension of monotonicity for real-valued signals. We show how to compute efficiently the degree of monotonicity.

A scalar-valued signal (or discrete function) is monotone if and only if the index set of values in any closed interval $[a, b]$ is a set of consecutive integers $[j, k]$: $p_i \in [a, b] \Leftrightarrow i \in [j, k]$. Equivalently, the values of the signal p_i never go down ($p_{i+1} \geq p_i$) or never go up ($p_{i+1} \leq p_i$). Another equivalent definition is given by the next proposition.

Proposition 5. *A scalar-valued signal p_i is monotone if and only if, for any 3 consecutive samples, p_i, p_{i+1}, p_{i+2} , the index set of the values contained in any closed interval $[a, b]$ is a set of consecutive integers $[j, k]$. Equivalently, the index set is a convex set under an appropriate definition of convexity.*

It is easy to extend this definition of monotonicity to the case of vector-valued signals. Unfortunately, a straightforward generalization, based on considering the set of indices of the values contained in any closed ball, would lead us to conclude that the only monotone vector-valued signals are on straight lines and never backtrack. It is not hard to realize no sensible filter could turn any vector-valued signal into a monotone signal. In order to obtain nontrivial results, we need to restrict the class of balls considered, as in the following definition.

Definition 1. *A vector-valued signal \mathbf{p}_i has a degree of monotonicity R if R is the largest value such that, considering only 3 consecutive samples, $\mathbf{p}_i, \mathbf{p}_{i+1}, \mathbf{p}_{i+2}$, the index set of the values contained in any closed ball B of radius at most R is a set of consecutive integers in $\{i, i + 1, i + 2\}$.*

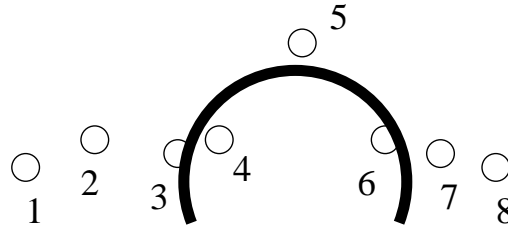


Fig. 2. Given the chain of data points shown, the degree of monotonicity is at most the size of the radius of the circle given in the picture: it contains points 4 and 6 but not point 5.

If the signal values are on a straight line with no backtracking, then the degree of monotonicity is ∞ , and the degree of monotonicity is always larger than 0 for finite signals. Fig. 2 gives an intuitive view of the degree of monotonicity. This measure of monotonicity is robust in the following sense.

Proposition 6. *If one point is omitted from a vector-valued signal, the degree of monotonicity cannot decrease.*

While this discrete definition is similar to the definition given for R -monotone curves, to allow efficient computation, we consider only sets of 3 consecutive samples, thus replacing a global problem by a local problem. If we lift the requirement that only 3 samples are considered, then a signal is R -monotone if and only if all subchains of length 3 are R -monotone. This suggests that the cost of checking global R -monotonicity grows in a cubic fashion with respect to the length of the signal which is unacceptable for most applications.

In practical applications, maximizing the degree of monotonicity R leads to useful chains. For example, noise tends to reduce R by creating sharp turns and local backtracking and a highly monotone curve (R large) is more likely to be noise-free. On the other hand, when reconstructing chains from unordered sets of points, as happens in computer vision, we often want to minimize sharp turns and backtracking. Therefore, solving for the chain maximizing R while passing through all available data points is a sensible “curve reconstruction” strategy.

As a prerequisite to computing the degree of monotonicity, we need a computationally effective way to compute the radius of the circle going through 3 points. Given $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \in \mathbb{R}^n$, we can compute the radius of the circle passing through them (denoted $\curvearrowright(\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3)$) by first computing $a = \|\mathbf{p}_1 - \mathbf{p}_2\|$, $b = \|\mathbf{p}_2 - \mathbf{p}_3\|$, $c = \|\mathbf{p}_1 - \mathbf{p}_3\|$, $\sigma = (a + b + c)/2$, and then we have the classical Heron’s formula for the radius of the circle:

$$R^{\text{outcircle}} = \frac{abc}{4\sqrt{\sigma(\sigma - a)(\sigma - b)(\sigma - c)}}$$

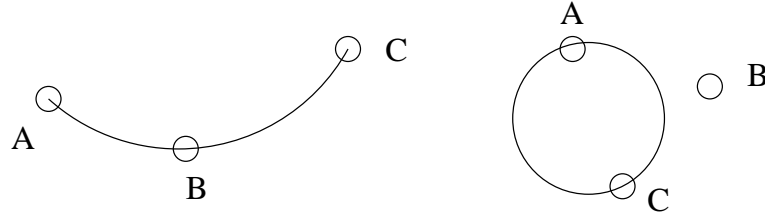


Fig. 3. Given a chain of 3 data points, we give two cases: (left) the angle $\angle(ABC) > \pi/2$ so we compute the radius of the circle going through ABC , otherwise (right), we compute half the distance between A and C .

whenever $a, b, c > 0$.

The next theorem gives us a way to compute the (local) degree of monotonicity for any 3 points, to compute the degree of monotonicity of an entire signal simply requires, by definition, to take the **minimum** of the result for all consecutive 3 points. The theorem essentially says that if $\angle(\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3) < \pi/2$, the degree of monotonicity is then half the distance between \mathbf{p}_1 and \mathbf{p}_3 , and otherwise, it is $R^{\text{outcircle}}$ (see Fig. 3). To see that this local form of monotonicity is distinct from the global form suggested earlier, consider a chain in the form of a figure “8.”

Theorem 1. *The degree of monotonicity for the sequence $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ is*

$$R := \begin{cases} \frac{1}{2}c & \text{if } a^2 + b^2 > c^2 \\ R^{\text{outcircle}} & \text{otherwise} \end{cases}$$

where $a = \|\mathbf{p}_1 - \mathbf{p}_2\|$, $b = \|\mathbf{p}_2 - \mathbf{p}_3\|$, $c = \|\mathbf{p}_1 - \mathbf{p}_3\|$.

Proof: Consider the disk B_0 containing \mathbf{p}_1 and \mathbf{p}_3 , centered at $(\mathbf{p}_1 + \mathbf{p}_3)/2$ and having radius $d(\mathbf{p}_1, \mathbf{p}_3)/2$. The point \mathbf{p}_2 is outside the disk if and only if $\cos \angle(\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3) = \frac{a^2 + b^2 - c^2}{2ab}$ is positive. Thus, \mathbf{p}_2 is outside the disk if and only if $a^2 + b^2 - c^2 > 0$. Clearly $R = \text{radius}(B_0) = d(\mathbf{p}_1, \mathbf{p}_3)/2$. Next, suppose that \mathbf{p}_2 is in the disk B_0 . We have that any ball containing \mathbf{p}_1 and \mathbf{p}_3 but not \mathbf{p}_2 must be larger than $\text{radius}(B_0)$ since B_0 is the smallest ball containing both \mathbf{p}_1 and \mathbf{p}_3 . Now, suppose there is a (closed) ball of minimal radius R containing \mathbf{p}_1 and \mathbf{p}_3 , but not \mathbf{p}_2 . This implies a non-zero distance, $\delta > 0$, between B and \mathbf{p}_2 . We have that the center of the ball has to be away from the line formed by $\mathbf{p}_1, \mathbf{p}_3$: if not then it must be a ball containing B_0 . This means we can move the center of the ball slightly closer to \mathbf{p}_1 and \mathbf{p}_3 while reducing the radius just enough so that \mathbf{p}_2 remains outside the ball. By repeating this process, we show that $\delta = 0$, a contradiction. Hence, there is no (closed) ball of minimal radius R containing \mathbf{p}_1 and \mathbf{p}_3 , but not \mathbf{p}_2 . Hence $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ have a degree of monotonicity $R^{\text{outcircle}}$. \square

§5. Monotonicity and Sphere-Preserving Filters

In this section, we propose a SP filter which never decreases the degree of monotonicity of the signal. Given a signal \mathbf{p}_i , we consider recursive (IIR) filters of the form

$$\mathbf{p}'_i = f(\mathbf{p}'_{i-2}, \mathbf{p}'_{i-1}, \mathbf{p}_i, \mathbf{p}_{i+1}, \mathbf{p}_{i+1}).$$

To ease the notation, we write $A = \mathbf{p}'_{i-2}$, $B = \mathbf{p}'_{i-1}$, $X = \mathbf{p}_i$, $C = \mathbf{p}_{i+1}$, $D = \mathbf{p}_{i+1}$ so that the equation becomes $X' = f(A, B, X, C, D)$. Let $R(A, B, X, C, D)$ be the degree of monotonicity of A, B, X, C, D computed as $\min(R(A, B, X), R(B, X, C), R(X, C, D))$. The following proposition gives us a condition of f to increase the monotonicity of a vector-valued signal.

Proposition 7. *Given $X' = f(A, B, X, C, D)$, if f is such that the degree of monotonicity*

$$R(A, B, X', C, D) \geq R(A, B, X, C, D),$$

then the recursive filter

$$\mathbf{p}'_i = f(\mathbf{p}'_{i-2}, \mathbf{p}'_{i-1}, \mathbf{p}_i, \mathbf{p}_{i+1}, \mathbf{p}_{i+1})$$

never decreases the degree of monotonicity of a signal.

It seems that f should be chosen so that $R(A, B, X', C, D)$ is as large as possible. To maximize $R(A, B, X', C, D)$ with $X' = f(A, B, X, C, D)$, f should be either B or C . In other words, we improve monotonicity best when we make the sample X “virtually disappear.”

Proposition 8. *$R(A, B, X', C, D)$ is minimized when $X' = B$ or $X' = C$ and these choices are unique unless $\frown(ABC) = \frown(BCD)$ in which case any point on the arc of the circle between B and C inclusively qualifies.*

Fortunately, we can easily define a more interesting SP filter. Given an arc of a circle, denoted α , and a point X , we can project X on α by solving for the point closest X in α . The projection onto a circle can be determined easily using only linear algebra [13]. In the plane, start with equation $(x - r_1)^2 + (y - r_2)^2 = \rho^2$ and substitute 3 values of x, y , getting 3 equations. By pairwise subtraction, we can remove the unknown ρ^2 , and be left with linear system having 2 equations and 2 unknowns (the center of the circle). We apply this by first projecting on the circle and if the projected point does not belong to the given arc we move it to the closest point on the arc (an endpoint of the arc). Let us define X_1 to be the projection of X on the arc BC of the circle ABC , and define X_2 to be

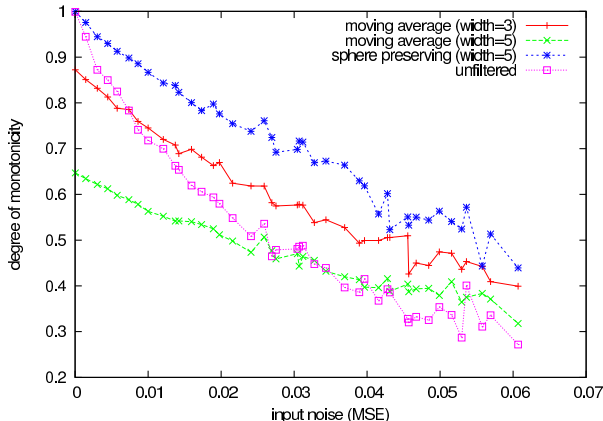


Fig. 4. The degree of monotonicity versus the absolute input noise level (MSE) over a synthetic data set generated from points on a unit circle. The SP filter outperforms MA when noise levels are low.

the projection of X on the arc BC of the circle BCD . Intuitively, either point X_1 or X_2 would make a good choice for X' . To ensure that the degree of monotonicity is never decreased, we set

$$f(A, B, X, C, D) = \arg \max_{X' \in \{X, X_1, X_2\}} R(A, B, X', C, D).$$

This function can be computed quickly and is sphere-preserving.

§6. Experimental Results

We generate a chain in the xy plane by regularly sampling a unit circle 3 times for a total of 30 samples. A MA filter with window width k averages each k consecutive data points. We add white noise to every point in the chain and we filter it using simple MA filters with window widths of 3 and 5 samples as well as with the SP filter of the previous section. Each test is repeated 10 times and we keep only the averages. Fig. 4 shows the degree of monotonicity versus the noise level (Mean Square Error) with the three smoothing filters and the unfiltered chain. The noise level ranges from none to over 0.05 (MSE) which corresponds roughly to a 5% noise-to-signal ratio. An example of filtering is given in Fig. 5.

§7. Discussion

In the unfiltered chain, the degree of monotonicity is inversely correlated with the noise level: the Pearson correlation is $p = -0.95$ (90%). The degree of monotonicity seems a good indicator of noise, which in particular suggests that a method for increasing the degree of monotonicity

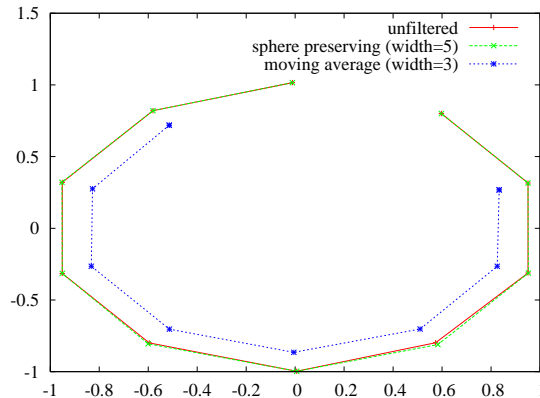


Fig. 5. Visual comparison of the SP filter with the MA filter for low noise levels and coarse sampling.

would also function as a good noise reduction technique. As required, the SP filter always increases the degree of monotonicity with respect to the unfiltered data. Simple MA filters **decrease the degree of monotonicity** when noise levels are low, and more aggressive filtering (window width of 5 versus 3) even more so. The result of aggressive lowpass filtering on the curvature of a chain is explained by Fig. 1. The relative performance of filters over chains can vary depending on the level of noise and the distance between the points: as noise levels increase, the SP filter is less competitive. The design of sphere-preserving filters optimally increasing the degree of monotonicity is an open problem.

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