Mathematical analogies: An engine for understanding the transformations of the boundaries between economics and physics

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ABSTRACT

To date, the relation between economics and physics has been analyzed through the influence of physics on economics. However, in the last years this relation has become more complex, because economic models are nowadays used in physics (minority game, GARCH model, etc.). The aim of this paper is to explain the origin of this new relation between physics and economics. It shows how mathematical analogies have contributed to progressively reshape the disciplinary boundaries of economics and physics, making them more permeable. It investigates three examples: Frisch’s “rocking horse” model (1933); the use of the Ising model for creating econophysics in the 1990s; the minority game, created by econophysicists in 1997 for solving an economic problem and nowadays used in physics. Our investigation demonstrates how mathematical analogies have started to influence the flow of thinking between physics and economics in both directions.

Keywords: Frisch; Le Corbeiller; mathematical analogies; economics and physics; history of business cycle theory; econophysics; minority game

JEL Classification: B16; B23; B26; B3; B4
**INTRODUCTION**

A number of historians and epistemologists of sciences and of economics have demonstrated how physics has deeply structured economics. Ménard (1978, 1989), Mirowski (1989), Schabas (1990), among others, have convincingly shown how, from the mid-19th century, authors such as Antoine-Augustin Cournot, William S. Jevons or Léon Walras have initiated the construction of economics as a “social physics”. Their work opened the way to further studies that extensively highlighted the contributions of physics to the development of various branches of contemporary economic analysis: business cycle analysis (Boumans 2004, Le Gall 1994, 1999, Morgan 1990), general equilibrium analysis (Weintraub 1985), monetary economics (Morgan 1997), or financial economics (Jovanovic and Le Gall 2001a, Jovanovic and Schinckus 2017, Sornette 2014). Beyond these specific branches, a large part of 20th century economic methodology was also inspired by physics: mathematical economics (Weintraub 2002), economic modeling (Le Gall 2002, Morgan and Morrison 1999) or econometrics (Le Gall 1994, 2007, Louçã 2007, Morgan 1990).

But should we consider definitive that the interrelations between economics and physics have gone from the latter to the former? Jovanovic, Mantegna, and Schinckus (2019) showed that during the last two decades a reciprocal movement occurred. Models and concepts from economics have contributed to developing several fields of physics. This result suggests that the relationship between the two disciplines is now more complex.

This paper aims at shedding some light on the new interactions between economics and physics. It shows how some changes in the nature of analogies have gradually transformed the boundaries between economics and physics and the dynamic between these disciplines. Specifically, we extend Israel’s (1996) analysis of “mathematical analogies”. These latter are based on a mathematical model that is employed as an autonomous container usable for describing phenomena from various fields. Our investigation demonstrates that these analogies have generated autonomous mathematical models built with concepts and formalisms from both economics and physics and then applied for studying phenomena in each of these two disciplines. Such autonomous mathematical models explain why no single-direction relation between the two disciplines now exists and why the boundaries between both fields can evolve from both directions.

The paper is organized as follows. Section I will briefly review the major characteristics of analogies and analogical reasoning. On this basis, we will discuss the historical change in analogies with the emergence of “mathematical analogies” in the 20th century, as documented by Giorgio Israel. We will discuss and extend his analysis in order to capture some evolutions in the relationship between economics and physics. Section II will use these elements for studying two examples of mathematical analogies related with economics and physics: Ragnar Frisch’s “rocking horse” model (1933), which revolutionized economic modelling and business cycles analysis, and the Ising model developed in the 1920s and that played a key role in the creation of econophysics in the 1990s. It is worth clarifying that we will not discuss the economic contents of Frisch’s model, we will discuss the way this model was based on mathematical analogies. On the basis of these examples, we show that while mathematical analogies lead to dropping the restriction that the source domain has to be a mechanical phenomenon, the initial mathematical formalism is still here rooted in one source domain only: physics. Then section III will study one of the most recent uses of mathematical analogies in economics: the minority game. Our analysis points out a singular evolution of these analogies that occurred here. Mathematical analogies lead to dropping an additional restriction: the formal similarities do not have only physical meaning. As we will show, this evolution explain the important changes in the inferences between economics and physics and how they now flow in both directions.
I. THE NATURE AND THE CHANGING FOUNDATIONS OF ANALOGIES

Analogies and analogical reasoning have been widely studied. Nevertheless, Israel’s work about the rise of “mathematical analogies” is less well-known, though it sheds light on important developments of sciences in the 20th century. This section presents and extends his work.

I.1 The nature of analogical reasoning

The usefulness of analogies and analogical reasoning in the production of scientific knowledge has been extensively analyzed (Bartha 2010, 2019, Norton 2018, chap. 4). Following Keynes (1921) and Hesse (1966), Bartha (2019, 1), who provided one of the clearest analyses, defines an analogy as “a comparison between two objects, or systems of objects, that highlights respects in which they are thought to be similar. Analogical reasoning is any type of thinking that relies upon an analogy”. Using Hesse’s (1966) terminology, the two objects which are compared belong respectively to a “source domain” and a “target domain”. Ménard (1989) offers a useful perspective on analogical reasoning by pointing out its dynamic process. Among other things, he shows how analogical reasoning creates knowledge in the “target domain” by considering this domain as a virgin territory.

Ménard (1989) conceived analogical reasoning as a three-step process. The first step is the “representation”, which consists in “circumscribing a relatively unknown territory” (Ménard 1989, 86). The scale of this new “territory” does not matter. At one extreme, it can concern a new phenomenon within a discipline whose frontiers are already defined. At another extreme, it can concern a broader field of research that could later become a discipline or, at least, a new branch of an existing discipline. In all cases, the analogy has as its goal an apprehension of an “unknown” territory (the “target domain”) on the basis of an already structured and organized knowledge devised in the “source domain” (Ménard 1989, 87). The second step: the analogy provides “a structure for classification and in this new way it creates similarities and differences where none existed before” (Ménard 1989, 87). It is a creative process that consists in exploring the target domain from the source domain by creating in the latter similarities and differences from what we know in the former. One could say that the target domain is ‘modelled’ from known features that come from the source domain. This work is achieved through a binary analysis: we find in the target domain characteristics/elements of the source domain; we do not find in the target domain characteristics/elements of the source domain. When similarities are more important than differences, this can impel a transfer from the source to the target. In that case, and this is the third step, the analogy can authorize “circulation and transfers” (Ménard 1989, 89) from the source to the target. Such transfers may concern concepts, tools, or methods.

I.2 The changing foundations of analogies through history

Despite the rich literature on analogies and analogical reasoning, something was unappreciated in the literature: the changing foundations of analogies through time. Israel in La mathématisation du réel (1996) investigated such a historical and philosophical change, leading him to point out a singular evolution. This author shows that this conceptual transformation arose through a shift, in his own words, between “mechanical analogies” and “mathematical analogies”.

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1 Ménard’s paper is the translation of a study written in French and published in 1981 in a two-volume book, Analogie et connaissance. This book is the sequel of interdisciplinary conferences on analogical reasoning held at the Collège de France.

2 Giorgio Israel (1945-2015), trained in mathematics, was a historian and philosopher of sciences. This book, written in French and only translated into Italian, remains little known by English speaking scholars.
Israel explained that from the 17th century to the 19th century, “mechanical analogies” flourished. They were the product of a philosophical worldview, according to which the natural world and the social world would be organized and ruled by a small set of similar causal laws of a deterministic, mathematical and mechanical nature. From this perspective, mechanical analogies had physics for source domain and used causal schemes afforded by mechanics for explaining phenomena. These laws contribute to forming a natural order ruling the universe. Thus,

“Science must offer a unitary and objective image of the Universe. Even if it is not possible to enclose the Universe in a single formula, the different parts of science and the theories that apply to different domains of phenomena must be at least linked together and coherent with each other. They must form a unitary construction, within which mechanics will always have the most important role” (Israel 1996, 18).

The aim of scientists was then to unveil these mechanical laws at work in the natural and the social worlds. For this reason, a number of 19th century economists, who introduced the mathematical language into the discipline, analyzed economic phenomena through the prism of mechanics. For example, Cournot (trained in mechanics) explained that markets and machines would be analogous ([1838] 1927, 9). He also claimed that economics was nothing but a “social physics” ([1872] 1973, 325). In the same vein, for Walras, the “procedure [in economics] is rigorously identical to that of the two of the most advanced and uncontested physico-mathematical sciences: rational mechanics and celestial mechanics” (Walras 1909, 316). Cournot, Jevons, Jules Regnault, Walras, Henry L. Moore, among others, intensively transferred to economics concepts, methods and laws from physics, explaining how economic phenomena would be analogous with natural phenomena. They supported such transfers by the hypothesis of a “single and unequivocal mathematical image of the reality” (Israel 1996, 11).

However, according to Israel, the worldview that supported the use of “mechanical analogies” progressively collapsed from the beginning of the 20th century. The shape and nature of analogies used in economics and other sciences changed significantly, becoming dominated by “mathematical analogies”. In both cases, the nature of analogical reasoning follows the same process: both kinds of analogies establish correspondences between elements from a source domain and other ones from a target domain. However, a crucial change occurred with mathematical analogies: the “autonomization” of mathematical models. More precisely, the analogical reasoning allows one to use a mathematical model as “an empty container, which can be filled with different contents” (Israel 1996, 38). Such mathematical schemes “unify different but isomorphic ‘realities’” (Israel 1996, 75). In other words, we can consider that in the process of making a “mathematical analogy”, we have a multiplicity of target domains and one single source domain, the mathematical model itself. Indeed, we do not have direct transfers from the source domain to the target domain, as it was the case with mechanical analogies. The mathematical model becomes an intermediary between the disciplines, allowing bi-directional correspondences (i.e. bijections) between physics and economics (or between any other fields). Armatte and Dahan clarified: “This modeling is characterized by the priority given to the formal description over the phenomenon, the non-linearity

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3 All translations from the French are ours, except when specified otherwise.

4 See Ménard (1978) and Le Gall (2007).

5 Translation by Mirowski and Cook (1990, 208).

6 On Regnault’s contributions to financial economics, see Jovanovic and Le Gall (2001a).
of the equations, the direct analogy between different domains, and the indirect verification (by simulation)” (2004, 248). In Israel’s analysis, these priorities justify the use of mathematical analogies.

Let us detail one of the emblematic illustrations of the early mathematical analogies intensively discussed by Israel: the model of “relaxation oscillations”. These oscillations owe their name to the fact that “a fast decrease, almost abrupt, follows a slow increase, then the cycle is repeated” (Israel 1996, 40). This model was designed by the Dutch physicist Balthasar Van Der Pol in 1926 in a specific field, electricity and radio applications. This author claimed that his model could simulate a large range of non-electric phenomena. This is a crucial point of his approach: this author was interested in the oscillations of the system, not in the functioning of the phenomena. In Van Der Pol’s perspective, his electrical model does not pretend to represent the mechanism of phenomena like the heart beating mechanism (one of his detailed example), but to simulate the behavior of the phenomena only. As he claimed, “the intimate resemblance of these phenomena, physically so dissimilar but mathematically analogous, cannot be denied” (Van Der Pol 1930, 255). He adds that “complicated systems, whose analytical solution would be difficult to obtain, can be solved experimentally in laboratories”. Further, from 1928 on, he explained that relaxation oscillations “are to be found in many realms of nature” (1940, 78). He afforded a list of phenomena that would obey them, including economic crises:

“A pneumatic hammer, the scratching noise of a knife on a plate, the waving of a flag in the wind, the humming noise sometimes made by a water tap, the squeaking of a door, … the periodic reoccurrence of epidemics and of economical crises, … the sleeping of flowers, the periodic reoccurrence of showers behind a depression, the shivering from cold, the menstruation and finally the beating of the heart” (Van Der Pol and Van der Mark 1928, 765-766).

In the same vein, Van Der Pol ended a conference (Eindhoven, August 1939) with the following words:

“Ladies and Gentlemen,
I hope to have it made clear to you, how different sciences are often governed by common mathematical laws and relations, and how a clearer and deeper insight into some phenomena may give us a vivid picture of what happens in other apparently totally unrelated phenomena in fields belonging to other sciences so that often mathematics bind together what at first sight seems to be utterly unrelated” (Van Der Pol 1940, 87).

In his view, the focus is put on the virtues of a mathematical model. His belief is based on two elements: a general mechanism (i.e. the relaxation oscillations) and the independency of the mathematical model regarding the phenomena at work in various domains that show oscillations.

What are the foundations of such a belief? As Israel explains, the mathematical model used to explore the nature of various phenomena does not originate in the philosophical worldview that dominated during the 18th and 19th centuries. The world is now considered as complex and no longer ruled by a small set of general, simple and mechanical causal laws. Given the supposed complexity of the world, scientists nowadays use mathematical models as preliminary exploratory frames aiming at reproducing, mimicking or simulating phenomena, rather than providing causal relations that could explain the functioning of the phenomena. As Israel clarifies,
“The mathematical modelling is a conceptual probe that is immersed in the reality, and not the mathematical image of the nature (...). Mathematical models are the sensors of this probe” (Israel 1996, 330).

Israel thus shows that “mathematical analogy” is based on the use of a mathematical model that could reproduce (via processes like calibration or parametrization) the behavior of a wide set of phenomena. It is important to note that the use of these analogies got rid of the causal schemes that were imposed by mechanical analogies: mathematical analogies are first engines for discovering regularities on the sole basis of reproduction.

I.3 Mathematical analogies: questions and perspectives

Israel’s analysis is very stimulating for understanding this historical change of the foundations of analogies. His analysis of mathematical analogies is mainly restricted to the 1930s, which he considered as the starting point of this movement. However, this movement can be traced back to 1919 and happened progressively (Ginoux 2017). Indeed in the 1930s some characteristics of the mathematical analogies were not established yet. For instance, Van Der Pol did not drop the causality, while Bourbaki did in 1948. However, this dropping is a major characteristic of mathematical analogies. Given this, it seems necessary to extend Israel’s description of mathematical analogies. From this perspective, we identify three relevant elements.

The first element: Israel did not point out that these analogies rose during the 1920s in the context of the “probabilistic revolution” (initiated by the birth of quantum theory). In fact, the transition from mechanical to mathematical analogies occurred in the context of modelling statistical ensembles where concepts of complexity and uncertainty play a major role. As Bartha (2019, section 4.2.3) reminds us,

“Steiner (1989, 1998) suggests that many of the analogies that played a major role in early twentieth-century physics count as “Pythagorean.” The term is meant to connote mathematical mysticism: a “Pythagorean” analogy is one founded on purely mathematical similarities that have no known physical basis at the time it is proposed. One example is Schrödinger’s use of analogy to “guess” the form of the relativistic wave equation. In Steiner’s view, Schrödinger’s reasoning relies upon manipulations and substitutions based on purely mathematical analogies”.

The second element: Israel pointed out that mathematical analogies are based on the possibility to link directly isomorphic phenomena by a mathematical model. But he did not emphasize on the field (i.e. the source domain) from which this mathematical model is created. In his examples, physics is the only source domain of the mathematical model: Van Der Pol’s model of relaxation oscillations was created in the field of electricity physics, and Volterra’s prey-predator model was based on mechanical thinking (and also the introduction of a probability that predator and prey meet). In retrospect, this perspective seems too restrictive: various sciences can be considered as source domains of the mathematical model (economics, biology, geometry...). A telling example is the mathematical physicist Enzo Tonti. In 1972, this author explains that “many physical theories show formal similarities due to the existence of a common mathematical structure. This structure

7 It is worth mentioning that Gingras (2015, 537) mentioned that “Philosophical reflections on the meaning of [mathematical] analogies in physics emerged only when mathematics came to play a central role in the construction of physical theories” and identified James Clerk Maxwell as the first author to raise this question in 1861.
is independent of the physical contents of the theory and can be found in classical, relativistic and quantum theories; for discrete and continuous systems” (Tonti 1972, 48). Tonti (1976) pointed out that these common mathematical properties derive from the fact that physical integral variables are naturally associated with the geometric elements of space (points, lines, surfaces and volumes) and time elements (instants and intervals). Relying on this observation, he provided an explanation of the analogies with physics: while physical phenomena have different physical meanings, they have in common the possibility of being associated with geometrical elements, allowing us to “construct a unique mathematical model for many physical theories” (Tonti 1976, 37). Thus, mathematical analogies can have a geometrical basis. However, it is worth noting that if a mathematical model allows direct mathematical analogies between isomorphic phenomena, then the model is independent of any domain and thus becomes an engine for isomorphism. Consequently, Ménard’s third step should disappear, and the mathematical model becomes in itself the new source domain.

The third element concerns the lack of explanatory power of these analogies that Israel repeatedly denounced. Remember that priority is given, among other things, to the formal description over the phenomenon and to the indirect verification by simulation. The similarities between the two domains (i.e. Ménard’s second step) become the statistical regularities themselves. According to Israel (1996, 50), the mathematical analogies would be evaluated on the ability of the mathematical model to reproduce statistically a class of phenomena; and this evaluation would be based on criteria such as “likelihood and utility”. However, this alleged instrumentalist nature of “mathematical analogies” is too limited. The use of “mathematical analogies” can be positively understood as attempts to discover forms of order in the context of a new worldview dominated by complexity and uncertainty. Mathematical analogies can help scientists to explore phenomena in the target domains, independently from any causal scheme given a priori, but this does not imply that scientists will try to provide theoretical explanation to the new phenomena. In this perspective, it seems that mathematical analogies are linked with a phenomenological approach: we can simulate the phenomena without explaining them. So, the theoretical explanation, if it occurs, will take place after the identification of empirical regularities. A telling example is Van Der Pol’s electrical model used for simulating the heart beating mechanism. More than 50 years later, Signorini and Di Bernardo (1998) showed that the heart beating mechanism is composed of two relaxation oscillations.

These elements help us to see how mathematical analogies alone have changed the relationship between economics and physics, as the next sections will show.

II. THE USES OF MATHEMATICAL ANALOGIES IN ECONOMICS
Several examples illustrate the use of mathematical analogies in economics. We have selected two examples for analyzing the changes they have generated in the boundaries of economics and physics.

II.1 Frisch’s rocking horse model: an early mathematical analogy in economics
As we emphasized, the earliest uses of “mathematical analogies” occurred in the context of the probabilistic revolution. In economics, this period was crucial: this is the moment when econometrics and macroeconomic modelling arose. In the early 1930s, when econometrics was becoming an institutionalized field, with Ragnar Frisch in the cockpit, the study of business cycles was on the forefront. Frisch’s 1933 article on the “rocking horse” is one of the first attempts to

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8 On the phenomenological approach in physics, see Cartwright (1983).
explain economic fluctuations through the use of a mathematical model. Frisch started his article by recalling the indebtedness of his model to the theory of oscillations:

“The majority of the economic oscillations (…) seem to be explained most plausibly as free oscillations (…). The most important feature of the free oscillations is that the length of the cycles and the tendency towards dampening are determined by the intrinsic structure of the swinging system, while the intensity (the amplitude) of the fluctuations is determined primarily by the exterior impulse” (1933, 1).

The rootedness of his model in the oscillation theory does not result from chance: when Frisch started working on this model from 1927 (Bjerkholt and Dupont-Kieffer 2009, Louça 2001, 29), a “new paradigm” had emerged in Europe with Van Der Pol’s work on relaxation oscillations (Petitgirard 2004, 433). With the introduction of relaxation oscillations in 1926, Van Der Pol’s non-linear oscillation model solved a problem on which physicists had been working on for several decades. At that time, France was a “crossroads” where various currents intersected and contributed to the development of a theory of non-linear oscillations (Ginoux 2017). One reason is that this new research was initiated by Henri Poincaré who established in 1908 that Van Der Pol’s equation admits a stable limit cycle type solution and therefore a periodic solution (Ginoux 2017, chap. 1). In the 1930s in France, research on oscillations was partly framed by Van der Pol’s “paradigm”, which explains why this author came regularly to France (Petitgirard 2004, 431). In 1930, the French physicist Philippe Le Corbeiller was assisting Van Der Pol at the Ecole Supérieure d’Electricité (Ginoux 2017, chap. 6). He became “the contact and relay for Van der Pol’s work in France, when he did not travel there himself” (Petitgirard 2004, fn 1254). Le Corbeiller, who was working on a general “theory of oscillations”, as he called it, continued to popularize Van Der Pol’s relaxation oscillations and their applications.

Several economists who worked on business cycles acknowledged the influence of Van Der Pol and Le Corbeiller. Goodwin is a telling example. As he explained, Le Corbeiller had a great influence on his work, in particular for stressing that nonlinearity was necessary for explaining how

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9 Louça (2007, 118-120) discussed some of Frisch’s positions from the perspective of Israel’s analysis. He compared for instance some key positions of Bourbaki (particularly causality) with Frisch’s ones, and concluded that Frisch’s approach was not compatible with the mathematical analogies. However, as section I explained, mathematical analogies evolved during the whole 20th Century. Remember that Van Der Pol did not drop the causality, while such a dropping is a major characteristic of mathematical analogies.

10 We are working on an extensive demonstration of the link between the rocking horse model and the theory of oscillations (Ginoux, Jovanovic, and Le Gall 2020). Providing such analysis here would take us far from our current topic.

11 Another reason for Van Der Pol regular presence in France is the connection with Poincaré’s work on self-sustained oscillations and limit cycles (Ginoux 2017). In other words, this research topic was extremely en vogue in France at this time. Moreover, Van Der Pol discovered Poincaré’s work thanks to Le Corbeiller.
macroeconomic oscillation could be a recurrent phenomenon (Goodwin 1951, 2). According to Goodwin, “Frisch misled a generation of investigators by resolving the problem [of explaining the continued existence of oscillations] with exogenous shocks, whereas already in the 1920s van der Pol had shown (as Frisch should have known) that a particular form of nonlinear theory was the appropriate solution. His solution leads to a limit cycle” (Goodwin 1990, 10)\textsuperscript{12}.

In his work, Frisch does not mention Van Der Pol or Le Corbeiller, while section 6 of his paper explicitly refers to “auto-maintained oscillations”, which are the general type of relaxation oscillations. In the early 1930s, at the time when his rocking horse was in gestation, Frisch knew these authors intimately\textsuperscript{13}. We can mention five proofs, among several others that we investigated in another paper (Ginoux, Jovanovic, and Le Gall 2020).

First, at the first Econometric Society meeting that held in September 1931 in Lausanne, Le Corbeiller presented his general theory of oscillations and explained how Van Der Pol’s model can be extended to economic crises and cycles\textsuperscript{14}. Frisch was a participant in the session and discussed Le Corbeiller’s ideas. In this discussion Frisch presented orally some early ideas that he developed later in his impulse-propagation model (Bjerkholt 2015, 1163).

Second, Frisch, as solo editor of 	extit{Econometrica}, accepted for publication in 1933 the paper Le Corbeiller presented in 1931. He claimed that “The problems of crises (…) is certainly one of the most difficult of Political Economy. In order to reach their solutions, it will be useful associate the resources of the theory of oscillations and those of economic theory” (1933, 328).

Third, in the same vein, Frisch accepted for publication in 1934 a note of Ludwig Hamburger recalling that he was working between 1928 and 1931 on the application of Van Der Pol’s relaxation oscillations to economic cycles by using an analogical reasoning. Hamburger was invited by Lucien March to publish his study in French in 1931\textsuperscript{15}. These publications demonstrate that Van Der Pol and Le Corbeiller’s works were already being used by economists and statisticians interested in business cycles when Frisch was thinking about his model.

Four, Frisch had a correspondence with Le Corbeiller and Hamburger\textsuperscript{16}. Frisch and Hamburger exchanged four letters between 1930 and 1933. After the Lausanne meeting, Frisch and Le Corbeiller exchanged at least six letters between 1932 and 1937 (some letters are missing).

Five, as Louça (2001, 32) has pointed out, in his correspondence with Schumpeter Frisch had integrated self-maintained oscillations into his analysis since June 1931.

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\textsuperscript{12} See also Zambelli (2007).

\textsuperscript{13} Le Corbeiller was an influential members of the Econometric Society (Louçã 2007, 112).

\textsuperscript{14} Regarding the correspondence between Frisch and François Divisia, it is probably Divisia who invited Le Corbeiller. Moreover, Divisia suggested to Frisch naming Volterra as a possible member for the Econometric Society.

\textsuperscript{15} Jovanovic and Le Gall (2001b) showed the key role of March in the development of business cycle analysis in France.

\textsuperscript{16} https://www.sv.uio.no/econ/om/tall-og-fakta/nobelprisvinnere/ragnar-frisch/correspondence/documents/Letters%20to%20Ragnar%20Frisch.%20Senders%5B1%5D.pdf.
The rootedness of Frisch’s model in the oscillation theory can be also observed in the title\textsuperscript{17}, the vocabulary\textsuperscript{18}, and more importantly in the structure of the article. Following the theory of oscillations discussed at this time, we know, and Frisch reminds us, that with a classical system (\textit{i.e.} without self-maintained oscillations) the oscillations are damped due to frictions. To solve this problem, an exogenous source is needed to maintain the oscillations. This is the goal of the second step of Frisch’s article (section 5 of his paper): its originality was to introduce exogenous random shocks in order to fit with the observed data\textsuperscript{19}.

But an exogenous source is not enough for maintaining the oscillations of the system. The oscillations of the variables of the system taken as a whole must not cancel each other out (which would create a monotonous system without oscillating in the extreme case). Frisch solves this problem first (section 3 of his paper). For doing that, he introduced two hypotheses that were formerly discussed by economists. Firstly, a causality between the variables of the system: one variable (the capital/production) transmits its oscillations to the other variables. This causality ensures that the oscillations of each variable will not dampen the oscillations of the system as a whole. Secondly, it is a time-dependency in the variables that creates oscillations.

Thanks to these two hypotheses, Frisch obtained a linear sustained oscillation model that can generate oscillations of a system conceived as purely abstract and general. But this model is still rough and it does not correctly mimic the reality. In order to have a model that fits with the observation, as Frisch clarified, a last step was needed. We have to “insert for the structural coefficients (…) numerical values that may in a rough way express the magnitudes which we would expect to find in actual economic life. At present I am only guessing very roughly at these parameters, but I believe that it will be possible by appropriate statistical methods to obtain more exact information about them. I think, indeed, that the statistical determination of such structural parameters will be one of the main objectives of the economic cycle analysis of the future” (Frisch 1933, 15). In other words, Frisch clarifies the need for calibrating the parameters of his model. This is a characteristic of his approach. At that time, such a statistical estimation process was a pure novelty in the field of economics. This point is also interesting because simulation and calibration are intimacy linked with oscillation modelling (Petitgirard 2004, 425-9). Indeed, such a methodology validates the mathematical analogy by identifying similitudes, which are statistical regularities, between the phenomena from different disciplines.

The “rocking horse” model shares many characteristics of a mathematical analogy approach. Based on the explanations given in section I, it is worth mentioning that we can use a mathematical model within a set of mechanical analogies. In this case, the mathematical model will be used for determining causal relations in the target field. This is not the case with mathematical analogies and with Van Der Pol’s nonlinear model. But Frisch couldn’t solve a nonlinear system at his time. He couldn’t investigate this avenue with his mathematical model. However, simulation, calibration, auto-maintained oscillations (section 6 of his paper), or his ex post explanations (sections 4, 5 and

\textsuperscript{17} The terms “impulse” and “propagation” are used in oscillation theory.

\textsuperscript{18} Frisch (1933) used the words “economic oscillations” whereas most economists at that time used the words “economic cycles”.

\textsuperscript{19} The introduction of such shocks was a first step toward the “probabilistic revolution” that fully occurred in econometrics with Haavelmo (1944).
are relevant elements of a mathematical analogy approach. In other words, although Frisch did not develop a nonlinear model like Van Der Pol, his conception is embedded in the mathematical analogy approach. This conclusion is also shared by Armatte and Dahan (2004, 252) who claimed that “the form of the model used by (…) Ragnar Frisch (...) is inspired by small oscillator models [and] directly derived from the practice of mathematical engineers, notably Van der Pol, who was in close touch with these econometricians (...). The reference to economic theory has faded away and only remains as the idea of a dynamic with delays between investment and production, between income and consumption. The ambition is clearly to build a small mechanism that can, for certain values of its parameters, simulate an economic mechanism, behave like an economic system”. We view this model as based on mathematical analogy and consider it as one of the first examples in economics.

**II.2 The Ising model: a mathematical analogy that gave birth to econophysics**

Our second example of the influence of mathematical analogies on boundaries between economics and physics concerns econophysics. The term *econophysics* refers to the extension of methods, models and concepts traditionally introduced and developed in the field of statistical and theoretical physics to the study of problems commonly considered to fall within the sphere of economics. The first use in print of the neologism “econophysics” occurred in a 1996 article by Stanley et al. (1996). In this seminal article, they claimed that “the analogy between economics and critical phenomena [described by the Ising model] is sufficiently strong that a similar story might evolve” (1996, 316). The Ising model, invented in 1920, was indeed crucial in the construction of econophysics (Jovanovic and Schinckus 2017, Schinckus 2018, Sornette 2014).

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20 The birth and the early development of econophysics is today well-known (Jovanovic and Schinckus 2013a, b, Kutner et al. 2019). Over the past two decades, econophysics has carved out a place in the scientific analysis of financial markets, macroeconomics, international economics, market microstructure, labor productivity, etc., providing new theoretical models, methods, and results (Aoyama et al. 2017, Gabaix 2009, Jovanovic and Schinckus 2017, Potters and Bouchaud 2003).

21 A critical phenomenon is a phenomenon for which the passage from one phase to another one is continuous. At the critical point, the system appears the same at all scales of analysis. This property is called “scale invariance,” which means that no matter how closely one looks, one sees the same properties. The dynamics of critical states can be characterized by a power law that deserves special attention, because this law is a key element in econophysics’ literature.

See Jovanovic and Schinckus (2017, chap. 3) for an extended definition of phase transitions and critical point in economic words.

22 The Ising model was invented by the physicist Wilhelm Lenz in 1920, and named after one of his PhD students Ernst Ising, solved it in 1925. It worth remembering that the Van Der Pol and Volterra models were created in the same decade.
Briefly, the Ising model consists of discrete variables that represent magnetic moments of atomic spins that can take one of two states, +1 (“up”) or −1 (“down”), the two states referring to the direction taken by the spins. The concept of spin characterizes the circular movement of particles (electrons, positrons, protons, etc.) implying that they have a specific rotation. There is no way to speed up or slow down the spin of an electron, but its direction can be changed, as modelled by the Ising model. The interesting point is that the direction of one spin directly influences the direction of its neighbor’s spins. This influence can be captured through a function of correlation that measures to what extent the behaviors of spins are correlated. The major idea of the Ising model is how to describe this interaction between particles’ spins. From this perspective, the spins are arranged in a graph, usually a lattice. The spin’s influence is measured by the distance over which the direction of one spin affects the direction of its neighbor’s spins. This distance is called the correlation length; it has an important function in the identification of critical phenomena. Indeed, the correlation length measures the distance over which the behavior of one microscopic variable is influenced by the behavior of another. Away from the critical point (at low temperatures), the spins point in the same direction. In such a situation, the thermal energy is too low to play a role; the direction of each spin depends only on its immediate neighbors making the correlation length finite. But at the critical point, when the temperature has been increased to reach the critical temperature, the situation is completely different. The spins no longer point in the same direction because the thermal energy dominates the whole system and the magnetization spin-spin vanishes. In this critical situation, spins point in no specific direction and follow a stochastic distribution. Moreover, each spin is now influenced by all other spins (not only its neighbors) whatever their distance. This situation is a particular configuration in which the correlation length is very important (it is considered to be infinite). At this critical state, the whole system appears to be in homogeneous configuration characterized by this infinite correlation length making the system scale invariant. Consequently, the spin system has the same physical properties whatever the scale length considered. It is worth mentioning that large variation correlation length appears to be ruled by a power-law, which is a key element of most of econophysics models.

The Ising model is a perfect illustration of the autonomous nature of mathematical model used as an empty container that characterize the mathematical analogies. Precisely, at the origin, it is a mathematical model of ferromagnetism used to study phase transitions and critical points. However, “it ‘possesses no ferromagnetic properties’” (Hughes 1999, 104). Its abstract and general structure permits its use to the study many other problems or phenomena characterized by phase transitions and critical points:

“The Ising model is employed in a variety of ways in the study of critical point phenomena. To recapitulate, Ising proposed it (…) as a model of ferromagnetism; subsequently it has been used to model, for example, liquid-vapour transitions and the behaviour of binary alloys. Each of these interpretations of the model is in terms of a specific example of critical point behavior (…). [T]he model also casts light on critical point behaviour in general. Likewise,

23 A power-law distribution is a special kind of probability distribution, such as \( p(x) = Cx^{-\alpha} \). Such distribution is leptokurtic, and it is widely used for studying variables that have extreme values. In economics, a well-known power law is the Pareto law. The main property of power laws is their scale invariance. Moreover, from the viewpoint of statistical physicists, power laws are synonymous with complex systems, which makes their study particularly valuable.
the pictures generated by computer simulation of the model’s behaviour illustrate (...) the whole field of scale-invariant properties” (Hughes 1999, 124-125).

For these reasons, the Ising model is considered as the simplest description of a system with a critical point; it has played a central role in the development of research on critical phenomena (Jovanovic and Schinkeus 2017, Sornette 2014). Moreover, the use of the Ising model is not restricted to statistical physics because “the specification of the model has no specific physical content” (Hughes 1999, 99); its content is mathematical. Therefore, as section I explained, while this model came from physics, it is independent of the underlying phenomenon studied, and it can be used to analyze any phenomena that share the same mathematical characteristics. In other words, the Ising model is a telling example of a mathematical model used in the process of mathematical analogies, which has allowed the transfers from physics to economics.

How did statistical physicists proceed? As Jovanovic and Schinkeus (2017, chap. 3) explained, statistical physicists adopt a phenomenological methodology. They identified phenomena with large numbers of interacting units whose microscopic behaviors could not be observed directly but which can generate observable macroscopic results. Therefore, modelers can look for statistical regularities often characterized by power laws, which are the main mathematical characteristic of critical phenomena, and also the main elements for justifying the application of statistical physics models, like the Ising model, to new phenomena and new target domains. As section I explained regarding the process of mathematical analogies, we have one mathematical model, which is the source domain and an empty container, and with a multiplicity of target domains. Moreover, it is worth saying that the process of identification used during this application of mathematical analogies is similar to Ménard’s second step: the target domain is "modelled" by the analogical reasoning. For instance, any work in econophysics advances empirical results to demonstrate that the phenomena studied are ruled by a distribution of variables or observables following a power law scaling, then varies (i.e. calibrate) the parameter $\alpha$ of the power laws until they obtain the best fit between the graph of the power law and the one describing the empirical data$^{24}$. However, power laws can visually be close to so-called exponential laws, and it is extremely difficult to distinguish between them$^{25}$. Therefore, this process of identifying a power law in the target domain is a telling illustration of how the domain source creates similarities in the target domain. At this stage, in the early works in econophysics, the mathematical analogies are similar to what we observed with Frisch: we have one source domain (i.e. physics) for the mathematical model.

To conclude this section, Frisch’s model and the Ising model illustrate the diffusion of the use of mathematical analogies for studying economic phenomena. In accordance with Israel’s analysis, these analogies drop the restriction that the source domain has to be mechanics or physical mechanics and its underlying causalities. However, the mathematical model used in the analogical process still has a physical meaning. Indeed, in the case of the oscillator, the formalism is mathematical, but there is a physical basis. It is similar with the Ising model. In other words, physics remains a justification for the use of mathematical analogies. However, as the next section will detail, the econophysicists’ approach has progressively transformed the mathematical analogies, changing the boundaries between economics and physics.

$^{24}$ In the same vein, econophysicists introduced truncation techniques for power laws in order to fit the observations of the target domain with their mathematical model (Jovanovic and Schinkeus 2017, chap. 3).

$^{25}$ In fact, many of the power laws econophysicists been trying to explain are not power laws at all (Clauset, Shalizi, and Newman 2009).
III. MATHEMATICAL ANALOGIES: ENGINES FOR CHANGING THE BOUNDARIES BETWEEN ECONOMICS AND PHYSICS

The last step of our analysis exposes the new interactions and avenues that emerge with econophysics. Jovanovic, Mantegna, and Schinckus (2019) showed that some models developed by econophysics are today being brought back into physics. They analyzed three illustrations of such transfers from economics to physics: the modelling of out of equilibrium processes, signal detection in multivariate systems and information process and aggregation in multi-agent physical systems. In our opinion, such transfers between the two disciplines were clearly allowed by mathematical analogies. A telling illustration of the way mathematical analogies create an opportunity to use an economic model for solving physical issues and show the evolution in mathematical analogies is the so-called “minority game”.

III.1 Minority game: new flow between physics and economics

Minority game was created mainly by two econophysicists, Challet and Zhang (1997). It is a stylized version of the “El Farol bar” problem. This is a well-known problem in game theory, originally introduced by an economist, W. Brian Arthur in 1994. At the 1994 AEA annual meeting, this author presented the following problem. N agents decide independently each week whether to go to a bar, named El Farol, that offers entertainment on a certain night. Each agent goes if he expects fewer than 60 people to show up or stays home if he expects more than 60 to go. The choices are unaffected by the previous visits; there is no collusion or prior communication among agents; the only information available is the numbers who came in the past weeks. No agent knows the model on which other agents base their predictions. Arthur (1994) concluded that in this context there is no deductively rational solution based on the theoretical hypotheses of economics: if all believe few will go, all will go, invalidating this would invalidate that belief; similarly, if all believe most will go, nobody will go, invalidating that belief. In other terms, the “El Farol bar” problem provides an illustrative example of the process of rational decision between two alternatives of a group of rational agents with the presence of negative externalities. In this setting, there is no self-fulfilling equilibrium and therefore in assuming fully rational use of the public information the system oscillates between states that is always frustrating for the agents.

Despite this evidence, Arthur decided to use computer experiments for searching if an equilibrium emerges. In his experiments, 100 agents use randomly use on strategy among k strategies. If the strategy does not work, the agent drops it for the future, otherwise he keeps it. His computer experiments showed a possible Nash equilibrium: on average 40 percent forecast above 60, 60 percent below 60. In other words, Arthur was able to show that a suboptimal (economic) equilibrium occurring at each time step around an a priori optimal allocation of the resource is reached by the system by hypothesizing a bounded rational inductive reasoning of the agents, but was not able to obtain a solution for such a situation “that is both evolutionary and complex” (1994, 411).

After demonstrating his economic equilibrium problem to economists, including economists specialized in game theory, Arthur had to admit that they did not know how to deal with it. He also noted that econophysicists found:

“economists didn’t quite know what to make of [my paper]. My colleague at Santa Fe, Per Bak, (…) saw the manuscript and began to fax it to his physics friends. The physics community took it up, and in the hands of Challet, Marsili and Zhang, it inspired something different than I expected –the Minority Game. El Farol emphasized (for me) the difficulties

In fact, Challet and Zhang (1997) formalized the “El Farol bar” by using a mathematical model developed by Caldarelli, Marsili and Zhang (1997) for studying the dynamics of financial market prices. This econophysic financial model draws also draws inspiration directly from the challenge Arthur addressed with the El Farol bar problem. Caldarelli et al. (1997, 479) observed that “from a physicist’s point of view, the market is an excellent example of self-organized systems: each agent decides according to his own perception of the events” and “the system reaches dynamically an equilibrium state characterized by fluctuations of any size, without the need of any parameter fine tuning or external driving”26. In other words, they were able to obtain a self-fulfilling equilibrium for the El Farol bar problem. To obtain such equilibrium, they introduced new hypotheses. Firstly, their reasoning is based on the typical financial economics equilibrium, the absence of arbitrage opportunity, which is not the typical economic market equilibrium. In this case, without new information, the price fluctuation dynamic should be stationary. Secondly, they analyze the El Farol Bar problem as an application of a Darwinist selection: at each time step, the agent with the smallest capital is eliminated and replaced by one with a new (random) strategy. From this perspective, the agents trade in a non-ending fight against each other in order to survive. Thirdly, they introduced the hypothesis of a power law to describe the distribution of the wealth of the traders. The numerical simulations of their model show close resemblance to the fluctuations of the Standard & Poor’s 500 index or to high frequency foreign exchange data. This result suggests that the stock market equilibrium is mainly due to the interaction among “speculators”, regardless of economic fundamentals. Moreover, and in accordance with Arthur’s problem, their model does not contain adaptive dynamics in the player’s strategies.

Challet and Zhang (1997) kept the hypothesis of a Darwinist selection for solving the El Farol problem: “the poor players are regularly weeded out from the game and new players are introduced to replace the eliminated one” (1997, 415). In other words, some agents are discouraged and give up going to the El Farol bar in the future. To keep a certain diversity, they also introduce a mutation possibility in cloning. They also allow one of the strategies of the best player to be replaced by a new one. With these additional hypotheses, they observe that this population is capable of “learning”. The learning process is demonstrated by their simulations. Specifically, they show that if we increase the information available used by the agents (i.e. number of past results about the numbers who came in the past weeks), the volatility decreases, meaning that the agents make the right choice more often. Moreover, agents who use less information underperform compare to the agent who use more information. “Remarkable is that each player is by definition selfish, not considerate to fellow players, yet somehow they manage to better somewhat share the limited available resources” (Challet and Zhang 1997, 409). “The players manage to defy entropy, in other words to get themselves organised to occupy less unlikely configurations” (1997, 412).

The Challet and Zhang (1997) and Caldarelli, Marsili and Zhang (1997) models use many key concepts of statistical physics, like entropy, power laws, “Self-organized criticality”, etc. The minority game is therefore a telling example how econophysics provides a solution to a problem that was originally conceived of in economics and was not tractable in terms of classic game theory with classic economic hypotheses.

26 “Self-organized criticality” is another key econophysics’ concept (Jovanovic and Schinckus 2017, chap. 5) Self-organized criticality is the property of dynamic systems that naturally self-organize into a critical point while obeying a power-law.
The minority game has another interesting feature. While it was created in order to analyze an economic problem, this model has been then applied back to physics and some related fields. It has been used for instance in radio engineering and computer science in order to improve wireless networks (Mähönen and Petrova 2008) or to improve coordination in wireless sensor networks (Galstyan, Krishnamachari, and Lerman 2004). In computer science, the minority game is used to improve the heterogeneous Delay Tolerant Networks (Sidi et al. 2013). According to Mähönen and Petrova (2008, 100), it could also be applied for studying the behaviors of flocks of birds. In other words, the minority game is an empty container that is applied to various phenomena totally independent of the economic phenomena in which it originated. It is worth noting that, in contrast to Van Der Pol or Le Corbeiller, Chalet and Zhang did not feel it necessary to give a list of possible applications. To us, this is due to the fact that creating mathematical models using a mathematical analogy is today a common practice. This observation is compatible with Claveau (2019): articles published in economics journals cited less and less articles published in mathematics journals, suggesting that mathematical models used by economists are more and more created within economics.

III.2 Minority game: new form of mathematical analogies

Our investigation shows that the minority game illustrates a noticeable change in the mathematical analogies. This mathematical model has been developed by borrowing simultaneously to economics and physics\(^\text{27}\). We noted that it integrates many key concepts of physics. However, it also integrates many key concepts of economics, such as the absence of arbitrage opportunity, rationality, cooperative games, and externalities. In other words, this mathematical model combines concepts and formalisms from economics and physics, which are both its source domains. In contrast, mathematical models saw in section II have only one source domain. However, they still have multiple target domains.

In our opinion, the institutional position of econophysics provides important clues for this transformation. Econophysics has a singular institutional position: outside economics and in the shadow of physics (Gingras and Schinckus 2012, Jovanovic and Schinckus 2017, chap. 4). This position reflects an autonomy of the field and has created the opportunity for developing new hypotheses and models outside physics and economics (Jovanovic and Schinckus 2017). But it also reflects a shift in the use of the mathematical analogies. The mathematical models econophysicists have developed become more autonomous, and have permitted transfers to both economics and physics. In other terms, there is not one source domain anymore, but two source domains (i.e. physics and economics) for the mathematical model and several target domains. This result is compatible with some changes in econophysicists’ perspective. Today they defends a “mutual fertilization”\(^\text{28}\) between both disciplines rather than a unidirectional influence of physics on economics, as it was common in the past. This position is compatible with the evolution of the discipline, as Jovanovic and Schinckus (2013c, 2017) have pointed out: it is becoming increasingly a transdisciplinary one. To conclude, these new perspectives are in line with the progressive change we pointed out: from mechanical analogies —restricted to one source domain that concern mechanical phenomena— to mathematical analogies that make room for the kinds of two-way inferences between economics and physics (i.e. as an isomorphism), because the emphasis is on the reproduction of phenomena rather than the causal explanation.

\(^{27}\) We detailed the example on the minority game. However, it is not the only one. For instance, physicists nowadays are using GARCH models (Modarres and Ouarda 2014).

\(^{28}\) This expression is borrowed from Sornette (2014, 1).
CONCLUDING REMARKS

New relations between economics and physics are emerging. Our investigation has showed how "mathematical analogies" have changed, and how they have contributed to transforming the disciplinary boundaries between economics and physics. These boundaries were drawn before the 20th century, while mathematical analogies emerged during the 20th century. As we have showed, the transformation of economics’ boundaries since the 1930s is directly concerned with the main characteristics of mathematical analogies. The latter are based on an autonomous mathematical structure that is fueled by concepts and formalisms that come from both disciplines. Our investigation does not pretend to be exhaustive, but it offers new perspectives on some of the evolutions of economics boundaries.

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