Mathematical analogies: An engine for understanding the transfers between economics and physics

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ABSTRACT

The influence of physics on economics has been largely analyzed; the opposite influence also exists even if it has been less studied. In the last decades the relation between these two disciplines has increased. Economic models are more often used in physics (minority game, GARCH model, etc.). The aim of this paper is to explore the role of mathematical analogies in the evolution of the relation between physics and economics. We show how these analogies have contributed to make the disciplinary boundaries of economics and physics more permeable. We investigate three examples: Frisch's PPIP model (1933); the use of the Ising model for creating econophysics in the 1990s; the minority game, created by econophysicists in 1997 for solving an economic problem and nowadays used in physics.

Keywords: Ragnar Frisch; Le Corbeiller; mathematical analogies; economics and physics; history of business cycle theory; econophysics; minority game

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1. INTRODUCTION

As Davis and Jovanovic (2021) reminded us, the study of the disciplinary boundaries of economics is an important subject of research for the history of economics, though it also is an inherently complex and often controversial subject. It challenges historians of economics to put the history of economics on a wider history of science map. In addition, the subject raises difficult philosophical issues. Economics' disciplinary boundaries may appear natural and inevitable, even though they are always at some level inherently artificial and constructed. To illustrate, consider how ideas have moved from economics to physics and back to economics. Thus, one scholar may focus on how Louis Bachelier, in discovering Brownian motion in 1900, applied the idea to financial markets, only to have it move back into physics later as well, and then also back again into economics (Rosser 2018). However, while Bachelier used financial data only once and for the purpose of proving his demonstration in mathematics (*i.e.* the equivalence between the results in probability obtained in discrete time and in continuous time, which led him to discover the Brownian motion), he was a mathematician who only published in journals of mathematics (Jovanovic 2012). For one scholar the use of financial data is enough to label Bachelier's work as economics, while for another the mathematical goal of Bachelier's demonstration is more relevant and sufficient to label his work as mathematics.

Several historians and epistemologists of sciences and of economics have investigated how physics has deeply structured economics. Ménard (1978; 1989), Mirowski (1989), Schabas (1990), among others, have convincingly shown how, from the mid-nineteenth century, authors such as Antoine-Augustin Cournot, William S. Jevons or Léon Walras have initiated the construction of economics as a "social physics." Their work opened the way to further studies that extensively highlighted the contributions of physics to the development of various branches of contemporary economic analysis: business cycle analysis (Le Gall 1994; 1999; Morgan 1990), general equilibrium analysis (Weintraub 1985), monetary economics (Morgan 1997), and financial economics (Jovanovic and Le Gall 2001; Jovanovic and Schinckus 2017; Sornette 2014). Beyond these specific branches, a large part of twentieth-century economic methodology was also inspired by physics: mathematical economics (Weintraub 2002), economic modeling (Le Gall 2002; Morgan and Morrison 1999) and econometrics (Le Gall 1994; 2007; Louçã 2007; Morgan 1990).

The influence of physics on economics is not surprising given that "Physics is the discipline that has provided most mathematical methods and templates for interdisciplinary transfer" (Knuuttila and Loettgers 2016, p. 378). Paul Humphreys (2002, p. S5) also mentioned the borrowing of techniques from physics to economics. The influence of economics on physics also exists, although it has been less analyzed. Indeed, some concepts developed in economics or for studying economic phenomena have been used after in physics. Pareto law is a well-known example (Rosser 2021). Jovanovic, Mantegna, and Schinckus (2019) showed that during the last two decades models and concepts from economics have contributed to developing several fields of physics. This observation suggests that the relationship between the two disciplines has become more complex, an observation also shared by Sornette (2014) and Rosser (2018).

This paper aims at shedding some light on the interactions between economics and physics from a historical perspective. It shows how some changes in the nature of analogies have made the boundaries between economics and physics more permeable. Specifically, we extend Israel's (1996) analysis of "mathematical analogies". This latter is based on a mathematical model that is employed as an autonomous container usable for describing phenomena from various fields. Our investigation demonstrates that these analogies have generated autonomous mathematical models built with concepts and formalisms from both economics and physics and then applied for studying phenomena in each of these two disciplines. Such autonomous mathematical models contribute to explain why no systematic single-direction relation between the two disciplines exists and why the boundaries between the two fields could evolve from both directions. Israel is not the only author to have analyzed such a transformation. Humphreys (2002) and Knuuttila and Loettgers (2016), among some others, also pointed out that some models and analogies have changed transfers between disciplines. Some of these authors also referred to Israel's analysis on mathematical analogies.

The paper is organized as follows. Section 2 briefly reviews the major characteristics of analogies and analogical reasoning. For the purpose of our analysis, we mainly focus on the dynamic evolution and process that have influenced analogical transfers between economics and physics. From this perspective, we discuss the historical change in analogies with the emergence of "mathematical analogies" in the twentieth century, as documented by Giorgio Israel. We discuss and extend his analysis in order to capture some examples in the relationship between economics and physics. Section 3 uses these elements for studying two examples of mathematical analogies related to economics and physics: Ragnar Frisch's PPIP[1] model (1933), which revolutionized economic modeling and business cycle analysis, and the Ising model developed in the 1920s and that played a key role in the creation of econophysics in the 1990s. It is worth clarifying that we will not discuss the economic contents of Frisch's model, we discuss the way this model was based on mathematical analogies. On the basis of these examples, we show that while mathematical analogies lead to dropping the restriction that the source domain has to be a mechanical phenomenon, the initial mathematical formalism is still here rooted in one source domain only: physics. Then section 4 studies one of the most recent uses of mathematical analogies in economics: the minority game. Our analysis points out a singular evolution of these analogies that occurred here. Mathematical analogies lead to dropping an additional restriction: the formal similarities do not have only physical meaning. As we show, this evolution explains the changes in the inferences between economics and physics.

2. THE NATURE AND THE CHANGING FOUNDATIONS OF ANALOGIES

Analogies and analogical reasoning have been widely studied. Nevertheless, Israel's work about the rise of "mathematical analogies" is less well-known, though it sheds light on developments of the interactions between scientific disciplines in the twentieth century. This section presents and extends his work.

2.1. The nature of analogical reasoning

The usefulness of analogies and analogical reasoning in the production of scientific knowledge has been extensively analyzed (Bartha 2010; 2019; Knuuttila and Loettgers 2014; Nersessian 2002; Norton 2018, p. chap. 4). Following Keynes (1921) and Hesse (1966), Bartha (2019, p. 1), who provided one of the clearest analyses, defines an *analogy* as "a comparison between two objects, or systems of objects, that highlights respects in which they are thought to be similar. *Analogical reasoning* is any type of thinking that relies upon an analogy." Using Hesse's (1966) terminology, the two objects which are compared belong respectively to a "source domain" and a "target domain." Ménard (1989) offers a useful perspective on analogical reasoning by pointing out its dynamic process.[2] Among other things, he shows how analogical reasoning creates knowledge in the "target domain" by considering this domain as a virgin territory.

[1] PPIP: Propagation Problems and Impulse Problems.

[2] Ménard's paper is the translation of a study written in French and published in 1981 in a two-volume book, *Analogie et connaissance*. This book is the sequel of interdisciplinary conferences on analogical reasoning held at the Collège de France.

Ménard (1989) conceived analogical reasoning as a three-step process. The first step is the "representation," which consists in "circumscribing a relatively unknown territory" (Ménard 1989, p. 86). The scale of this new "territory" does not matter. At one extreme, it can concern a new phenomenon within a discipline whose frontiers are already defined. At another extreme, it can concern a broader field of research that could later become a discipline or, at least, a new branch of an existing discipline. In all cases, the analogy has as its goal an apprehension of an "unknown" territory (the "target domain") on the basis of an already structured and organized knowledge devised in the "source domain" (Ménard 1989, p. 87). In a second step, the analogy provides "a structure for classification and in this new way it creates similarities and differences where none existed before" (Ménard 1989, p. 87). It is a creative process that consists in exploring the target domain from the source domain by creating in the former similarities and differences from what we know in the latter. One could say that the target domain is 'modeled' from known features that come from the source domain. This work is achieved through a binary analysis: on the one hand, we find in the target domain characteristics/elements of the source domain; on the other hand, we do not find in the target domain characteristics/elements of the source domain. When similarities are more important than differences, this can impel a transfer from the source to the target. In that case, and this is the third step, the analogy can authorize "circulation and transfers" (Ménard 1989, p. 89) from the source to the target. Such transfers may concern concepts, tools, or methods.

2.2. *The changing foundations of analogies through history*

Despite the rich literature on analogies and analogical reasoning, an issue was unappreciated: the changing foundations of analogies through time. Israel in *La mathématisation du réel* (1996) investigated such a historical and philosophical change, leading him to point out a singular evolution.[3] This author shows that this conceptual transformation arose through a shift, in his own words, between "mechanical analogies" and "mathematical analogies."

Israel explained that from the seventeenth century to the nineteenth century, "mechanical analogies" flourished. They were the product of a philosophical worldview, according to which the natural world and the social world would be organized and ruled by a small set of similar causal laws of a deterministic, mathematical and mechanical nature. From this perspective, mechanical analogies had physics for the *source domain* and used causal schemes afforded by mechanics for explaining phenomena. These laws contribute to forming a natural order to rule the universe. Thus,

> Science must offer a unitary and objective image of the Universe. Even if it is not possible to enclose the Universe in a single formula, the different parts of science and the theories that apply to different domains of phenomena must be at least linked together and coherent with each other. They must form a unitary construction, within which mechanics will always have the most important role. (Israel 1996, p. 18)[4]

The aim of scientists was then to unveil these mechanical laws at work in the natural and the social worlds. For this reason, a number of nineteenth-century economists, who introduced the mathematical language into the discipline, analyzed economic phenomena through the prism of mechanics. For example, Cournot (trained in mechanics) explained that markets and machines would be analogous ([1838] 1927, p. 9).[5] He also claimed that economics was nothing but a "*social*

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[3] Giorgio Israel (1945-2015), trained in mathematics, was a historian and philosopher of sciences. This book, written in French and only translated into Italian, remains little known by English speaking scholars.
[4] All translations from the French are ours, except when specified otherwise.
[5] See Ménard (1978) and Le Gall (2007).

physics" ([1872] 1973, p. 325). In the same vein, for Walras, the "procedure [in economics] is rigorously identical to that of the two of the most advanced and uncontested physico-mathematical sciences: *rational mechanics* and *celestial mechanics"* (Walras 1909, p. 316).[6] Cournot, Jevons, Jules Regnault[7], Walras, Henry L. Moore, among others, intensively transferred to economics' concepts, methods and laws from physics, explaining how economic phenomena would be analogous with natural phenomena. They supported such transfers by the hypothesis of a "single and unequivocal mathematical image of the reality" (Israel 1996, p. 11).

However, according to Israel, the worldview that supported the use of "mechanical analogies" progressively collapsed from the beginning of the twentieth century. The shape and nature of analogies used in economics and in other sciences changed significantly, becoming dominated by "mathematical analogies." In both cases, the nature of analogical reasoning follows the same process: both kinds of analogies establish correspondences between elements from a source domain and other ones from a target domain. However, a crucial change occurred with mathematical analogies: the "autonomization" of mathematical models. More precisely, the analogical reasoning allows one to use a mathematical model as "an empty container, which can be filled with different contents" (Israel 1996, p. 38). Such mathematical schemes "unify different but isomorphic 'realities'" (Israel 1996, p. 75). In other words, in Israel's perspective, we can consider that in the process of designing a "mathematical analogy," we have a multiplicity of target domains and one single source domain, the mathematical model itself. Humphreys' computational templates (part of what Humphreys calls computational models) seem rather close to Israel's mathematical analogies, in that both try to capture the way mathematical similarities drive the transfer of models across different disciplines. A computational template is relatively independent of any specific subject: "templates with different interpretations are not reinterpretations of the same model, but are different computational models entirely" (Humphreys 2002, p. S7). Humphreys also mentioned that computational models "can be an autonomous object of study" (2002, p. S10). The transfer of a modelling technique involves applying the template, not the model, to a new subject matter; there is, strictly speaking, no *model* transfer. "A key feature of computational templates is their ability to be applied across a number of different scientific disciplines" (2002, p. S3). Of course, this multiple applicability of formal descriptions formed the basis of analog computers that modelled mechanical systems with electronic systems, both systems being covered by the same computational template. However, contrary to mathematical analogies, computational models have been intensively developed thanks to "computer-assisted solutions" (2002, p. S5). As Knuuttila and Loettgers (2016, p. 397) explained, transfers between the disciplines allowed by the mathematical model are "not just based on its computational aspects." For this reason, these authors considered the concept of computational template introduced by Humphreys (2002) as too restrictive, and they introduced the concept of model template, which is a mixture between the concepts of Humphreys (computational template) and Israel (mathematical analogy). Indeed, there is a close proximity between mathematical analogy and model template. For this reason, the analysis of Knuuttila and Loettgers (2016, p. 379) shows several similarities with the analysis of Israel when they studied "the trajectory of one model within a multitude of fields." Indeed, we do not have direct transfers from the source domain to the target domain, as it was the case with mechanical analogies. As Israel (1996), Knuuttila and Loettgers (2016) identified that the mathematical model becomes an intermediary between the disciplines, allowing bi-directional correspondences (*i.e.* bijections) between physics and economics (or between any other fields). Armatte and Dahan clarified: "This modeling is characterized by the priority given to the formal description over the phenomenon, the non-linearity of the equations, the direct analogy between different domains, and the indirect verification (by simulation)" (2004, p. 248). In Israel's analysis, these priorities justify the use of

[6] Translation by Mirowski and Cook (1990, p. 208).
[7] On Regnault's contributions to financial economics, see Jovanovic and Le Gall (2001).

mathematical analogies. However, our article suggests that mathematical analogies have changed during the twentieth century: while we can have one source domain (physics as in Frisch's model) now we observe multiple source domains (as in the minority game model).

In order to analyze such a change, let us detail one of the emblematic illustrations of the early mathematical analogies that Israel intensively discussed: the model of "relaxation oscillations." These oscillations owe their name to the fact that "a fast decrease, almost abrupt, follows a slow increase, then the cycle is repeated" (Israel 1996, p. 40). This model was designed by the Dutch physicist Balthasar van der Pol in 1926 in a specific field, electricity and radio applications. This author claimed that his model could simulate a large range of non-electric phenomena. As Israel (1996) pointed out, this is a crucial point of his approach: he was interested in the oscillations of the system, not in the functioning of the phenomena. In Van der Pol's perspective, his electrical model does not pretend to represent the mechanism of phenomena like the heart beating mechanism (one of his detailed examples), but to simulate the behavior of the phenomena only. As he claimed, "the intimate resemblance of these phenomena, physically so dissimilar but mathematically analogous, cannot be denied" (Van der Pol 1930, p. 255). He adds that "complicated systems, whose analytical solution would be difficult to obtain, can be solved experimentally in laboratories." Further, from 1928 on, he explained that relaxation oscillations "are to be found in many realms of nature" (1940, p. 78). He afforded a list of heterogenous phenomena that would obey them, including economic crises (Van der Pol and Van der Mark 1928, pp. 765-66).

Based on such result, Van der Pol ended a conference (Eindhoven, August 1939) with the following words:

> Ladies and Gentlemen,
> I hope to have it made clear to you, how different sciences are often governed by common mathematical laws and relations, and how a clearer and deeper insight into some phenomena may give us a vivid picture of what happens in other apparently totally unrelated phenomena in fields belonging to other sciences so that often mathematics bind together what at first sight seems to be utterly unrelated (Van der Pol 1940, p. 87).

In his view, the focus is put on the virtues of a mathematical model. His belief is based on two elements: a general mechanism (*i.e.*, the relaxation oscillations) and the independency of the mathematical model regarding the phenomena at work in various domains that show oscillations.

According to Israel (1996), the mathematical model used to explore the nature of various phenomena does not originate in the philosophical worldview that dominated during the eighteenth and nineteenth centuries. The world is now considered as complex and no longer ruled by a small set of general, simple and mechanical causal laws. Given the supposed complexity of the world, scientists nowadays use mathematical models as preliminary exploratory frames aiming at reproducing, mimicking or simulating phenomena, rather than providing causal relations that could explain the functioning of the phenomena. As Israel clarifies,

> The mathematical modeling is a conceptual probe that is immersed in the reality, and not the mathematical image of the nature (…). Mathematical models are the sensors of this probe (Israel 1996, p. 330).

Israel thus shows that "mathematical analogy" is based on the use of a mathematical model that could reproduce the behavior of a wide set of phenomena. It is important to note that the use of these analogies got rid of the causal schemes that were imposed by mechanical analogies:

mathematical analogies are first engines for discovering regularities on the sole basis of reproduction.

2.3. Mathematical analogies: questions and perspectives

Israel's analysis is very stimulating for understanding this historical change of the foundations of transfers between disciplines through analogies. His analysis of mathematical analogies is mainly restricted to the 1930s, which he considered as the starting point of this movement. However, according to Ginoux (2017), this movement can be traced back to 1919 and happened progressively. Indeed, in the 1930s some characteristics of the mathematical analogies were not established yet. For instance, Van der Pol did not drop the causality, while Bourbaki did in 1948. However, this dropping is a major characteristic of mathematical analogies. Given this, it seems necessary to extend Israel's description of mathematical analogies. From this perspective, we identify three relevant elements.

The first element: Israel did not point out that these analogies rose during the 1920s. In fact, the transition from mechanical to mathematical analogies occurred in the context of modeling statistical ensembles where concepts of complexity and uncertainty play a major role. As Bartha (2019, p. section 4.2.3) reminds us,

> Steiner (1989, 1998) suggests that many of the analogies that played a major role in early twentieth-century physics count as "Pythagorean." The term is meant to connote mathematical mysticism: a "Pythagorean" analogy is one founded on purely mathematical similarities that have no known physical basis at the time it is proposed. One example is Schrödinger's use of analogy to "guess" the form of the relativistic wave equation. In Steiner's view, Schrödinger's reasoning relies upon manipulations and substitutions based on purely mathematical analogies.

In the same vein, Gingras (2015, p. 537) mentioned that "Philosophical reflections on the meaning of [mathematical] analogies in physics emerged only when mathematics came to play a central role in the construction of physical theories" and identified James Clerk Maxwell as the first author to raise this question in 1861.

The second element: Israel pointed out that mathematical analogies are based on the possibility to link directly isomorphic phenomena by a mathematical model. But he did not emphasize on the field (*i.e.,* the source domain) from which this mathematical model is created. In the examples detailed by Israel, physics is the only source domain of the mathematical model: Van der Pol's model of relaxation oscillations was created in the field of electricity physics, and Volterra's prey-predator model was based on mechanical thinking (and also the introduction of a probability that predator and prey meet).[8] In retrospect, this perspective seems too restrictive: various sciences can be considered as source domains of the mathematical model (economics, biology, geometry…). A telling example is the mathematical physicist Enzo Tonti. In 1972, this author explains that "many physical theories show formal similarities due to the existence of a common mathematical structure. This structure is independent of the physical contents of the theory and can be found in classical, relativistic and quantum theories; for discrete and continuous systems" (Tonti 1972, p. 48). Tonti (1976) pointed out that these common mathematical properties derive from the fact that physical integral variables are naturally associated with the geometric elements of space (points,

[8] Knuuttila and Loettgers (2016, p. 378) clarify that "Vito Volterra cast his theorizing on biological associations – the most famous example of which is the Lotka–Volterra model – in terms of "mathematical analogies" drawn from mechanics (Volterra 1901)".

lines, surfaces and volumes) and time elements (instants and intervals). Relying on this observation, he provided an explanation of the analogies with physics: while physical phenomena have different physical meanings, they have in common the possibility of being associated with geometrical elements, allowing us to "construct a unique mathematical model for many physical theories" (Tonti 1976, p. 37). Thus, mathematical analogies can have a geometrical basis. However, it is worth noting that if a mathematical model allows direct mathematical analogies between isomorphic phenomena, then the model is independent of any domain and thus becomes an engine for isomorphism. Consequently, Ménard's third step should disappear, and the mathematical model becomes in itself the new source domain.

The third element concerns the lack of explanatory power of these analogies that Israel repeatedly denounced. Remember that priority is given, among other things, to the formal description over the phenomenon and to the indirect verification by simulation. The similarities between the two domains (*i.e.* Ménard's second step) become the statistical regularities themselves (or the computational regularities in Humphreys (2002) perspective). According to Israel (1996, p. 50), the mathematical analogies would be evaluated on the ability of the mathematical model to reproduce statistically a class of phenomena; and this evaluation would be based on criteria such as "likelihood and utility." However, this alleged instrumentalist nature of "mathematical analogies" is too limited. The use of "mathematical analogies" can be positively understood as attempts to discover forms of order in the context of a new worldview dominated by complexity and uncertainty. Mathematical analogies can help scientists to explore phenomena in the target domains, independently from any causal scheme given *a priori*, but this does not imply that scientists look for theoretical explanation to the new phenomena. In this perspective, it seems that mathematical analogies are linked with a phenomenological approach: we can simulate the phenomena without explaining them.[9] So, the theoretical explanation, if it occurs, will take place after the identification of empirical regularities. A telling example is Van der Pol's electrical model used for simulating the heart beating mechanism. More than 50 years later, Signorini and Di Bernardo (1998) showed that the heart beating mechanism is composed of two relaxation oscillations.

These elements help us to see how mathematical analogies have changed the relationship between economics and physics, as the next sections show.

3. THE USES OF MATHEMATICAL ANALOGIES IN ECONOMICS

Several examples illustrate the use of mathematical analogies in economics. We have selected two examples for analyzing the changes they have generated in the boundaries of economics and physics.

3.1. Frisch's PPIP model: an early mathematical analogy in economics

In the early 1930s, when econometrics was becoming an institutionalized field, with Ragnar Frisch in the cockpit, the study of business cycles was on the forefront. Frisch's 1933 article on propagations and impulsions in economic cycles is one of the first attempts to explain economic fluctuations through the use of a mathematical model.[10] Frisch started his article by recalling the

[9] On the phenomenological approach in physics, see Cartwright (1983).

[10] Louça (2007, pp. 118-20) discussed some of Frisch's positions from the perspective of Israel's analysis. He compared for instance some key positions of Bourbaki (particularly causality) with Frisch's ones, and concluded that Frisch's approach was not compatible with the mathematical analogies. However, as section I explained, mathematical

indebtedness of his model to the theory of oscillations. The rootedness of Frisch's model in the oscillation theory does not result from chance: when Frisch started working on this model from 1927 (Bjerkholt and Dupont-Kieffer 2009), a "new paradigm" had emerged in Europe with Van Der Pol's work on relaxation oscillations (Petitgirard 2004, p. 433). Frisch participated to this new paradigm by creating a new kind of oscillation in his 1933 article, *forced damped oscillations* with "irregular forcing" (Ginoux and Jovanovic 2021)[11].

With the introduction of relaxation oscillations in 1926, Van der Pol's non-linear oscillation model solved a problem on which physicists had been working on for several decades. At that time, France was a "crossroads" where various currents intersected and contributed to the development of a theory of non-linear oscillations (Ginoux 2017). One reason is that this new research was initiated by Henri Poincaré who established in 1908 that Van der Pol's equation admits a stable limit cycle type solution and therefore a periodic solution (Ginoux 2017, p. chap. 1). In the 1930s in France, research on oscillations was partly framed by Van der Pol's "paradigm," which explains why this author came regularly to France (Petitgirard 2004, p. 431).[12] In 1930, the French physicist Philippe Le Corbeiller was assisting Van der Pol at the Ecole Supérieure d'Electricité (Ginoux 2017, p. chap. 6). He became "the contact and relay for Van der Pol's work in France, when he did not travel there himself" (Petitgirard 2004, pp. 431, fn 1254). Le Corbeiller, who was working on a general "theory of oscillations," as he called it, continued to popularize Van der Pol's relaxation oscillations and their applications[13].

In his work, Frisch does not mention Van der Pol, Le Corbeiller, or Ludwig Hamburger[14] while section 6 of his paper explicitly refers to "auto-maintained oscillations," which are the general type of relaxation oscillations. In the early 1930s, at the time when his PPIP model was in gestation, Frisch was interested in this approach (Ginoux and Jovanovic 2021; Jovanovic and Ginoux 2020). As Louçã (2001, p. 32) has pointed out, in his correspondence with Schumpeter Frisch had integrated self-maintained oscillations into his analysis since June 1931. However, Frisch proposed another king of model: a model of *forced damped oscillations* with "irregular forcing" that he called "erratic shocks" or "impulses" which compensate the damping and maintain the oscillations. Such class of oscillations did not exist in 1933, Some of the most important mathematicians and physicists who were developing the theory of oscillations at this time had been influenced by Frisch's article (Ginoux and Jovanovic 2021). Precisely, in 1937, Andronov and Khaikin (1937) dedicated their chapter the "Theory of the clock" to this new type of oscillation; they renamed Frisch's new oscillations *impulse-excited oscillations*. In the same vein, the French physicist Yves

analogies evolved during the whole twentieth century. Remember that Van der Pol did not drop the causality, while such a dropping is a major characteristic of mathematical analogies.

[11] However, it is worth reminding that at his time oscillation theory was still in its infancy and its formalism was not completely fixed. Therefore, the definitions and terminologies used by Frisch are confusing and have generated many misinterpretations in the literature (Jovanovic and Ginoux 2020).

[12] Another reason for Van der Pol regular presence in France is the connection with Poincaré's work on self-sustained oscillations and limit cycles (Ginoux 2017). In other words, this research topic was extremely *en vogue* in France at this time. Moreover, Van der Pol discovered Poincaré's work thanks to Le Corbeiller.

[13] Several economists who worked on business cycles acknowledged the influence of Van der Pol and Le Corbeiller. Richard M. Goodwin is a telling example. As he explained, Le Corbeiller had a great influence on his work, in particular for stressing that nonlinearity was necessary for explaining how macroeconomic oscillation could be a recurrent phenomenon (Goodwin 1951, p. 2). During the World War II, when he was in Harvard, Goodwing taught physics. It was during this period that he met Le Corbeiller in his office and asked him "whether he would teach [him] the theory of nonlinear dynamics. He, then, literally took [him] "by the hand" and taught [him] nonlinear dynamics" (Velupillai 2017, p. fn. 4). See also Velupillai (1998).

[14] Hamburger was working between 1928 and 1931 on the application of Van der Pol's relaxation oscillations to economic cycles by using an analogical reasoning.

Rocard (1941), who contributed to develop the theory of oscillations, deeply analyzed the simplified version of the PPIP model proposed by Kalecki (1935)[15].

The PPIP model shares many characteristics of a mathematical analogy approach. Based on the explanations given in section I, it is worth mentioning that we can use a mathematical model within a set of mechanical analogies. In this case, the mathematical model will be used for determining causal relations in the target field. This is not the case with mathematical analogies and with Van der Pol's nonlinear model or with Frisch's new oscillations. His ex-post explanations (sections 4, 5 and 6) and his PPIP model are relevant elements of a mathematical analogy approach. In other words, although Frisch did not develop a nonlinear model like Van Der Pol, his conception is embedded in the mathematical analogy approach. This conclusion is also shared by Armatte and Dahan (2004, p. 252) who claimed that "the form of the model used by (…) Ragnar Frisch (...) is inspired by small oscillator models [and] directly derived from the practice of mathematical engineers, notably Van der Pol, who was in close touch with these econometricians (...). The reference to economic theory has faded away and only remains as the idea of a dynamic with delays between investment and production, between income and consumption. The ambition is clearly to build a small mechanism that can, for certain values of its parameters, simulate an economic mechanism, behave like an economic system." We view this model as based on mathematical analogy and consider it as one of the first examples in economics.

3.2. *The Ising model: a mathematical analogy at the center of the birth of econophysics*

Our second example of the influence of mathematical analogies on boundaries between economics and physics concerns econophysics.[16] The term *econophysics* refers to the extension of methods, models and concepts traditionally introduced and developed in the field of statistical and theoretical physics to the study of problems commonly considered to fall within the sphere of economics. The first use in print of the neologism "econophysics" occurred in a 1996 article by Stanley et al. (1996). In this seminal article, they claimed that "the analogy between economics and critical phenomena [described by the Ising model] is sufficiently strong that a similar story might evolve" (1996, p. 316).[17] The Ising model, invented in 1920, was indeed crucial in the construction of econophysics (Jovanovic and Schinckus 2017; Schinckus 2018; Sornette 2014).[18]

Briefly, the Ising model consists of discrete variables that represent magnetic moments of atomic spins that can take one of two states, +1 ("up") or −1 ("down"), the two states referring to the direction taken by the spins. The concept of spin characterizes the circular movement of particles

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[15] Frisch's PPIP led to one of the most fruitful debates with another key econometrician, Michał Kalecki (1935). This debate started during the third E*conometric society* meeting held in Leyde in 1933 and was published in *Econometrica* from 1935 onwards.

[16] The birth and the early development of econophysics is today well-known (Jovanovic and Schinckus 2013; Kutner *et al.* 2019; Rosser 2006; 2018; Schinckus 2021). Over the past two decades, econophysics has carved out a place in the scientific analysis of financial markets, macroeconomics, international economics, market microstructure, labor productivity, etc., providing new theoretical models, methods, and results (Aoyama *et al.* 2017; Gabaix 2009; Jovanovic and Schinckus 2017; Potters and Bouchaud 2003).

[17] A critical phenomenon is a phenomenon for which the passage from one phase to another one is continuous. At the critical point, the system appears the same at all scales of analysis. This property is called "scale invariance," which means that no matter how closely one looks, one sees the same properties. The dynamics of critical states can be characterized by a power law that deserves special attention, because this law is a key element in econophysics' literature. See Jovanovic and Schinckus (2017, p. chap. 3) for an extended definition of phase transitions and critical point in economic terms.

[18] The Ising model was invented by the physicist Wilhelm Lenz in 1920, and named after one of his PhD students Ernst Ising, solved it in 1925.

(electrons, positrons, protons, etc.) implying that they have a specific rotation. There is no way to speed up or slow down the spin of an electron, but its direction can be changed, as modeled by the Ising model. The interesting point is that the direction of one spin directly influences the direction of its neighbor's spins. This influence can be captured through a function of correlation that measures to what extent the behaviors of spins are correlated. The major idea of the Ising model is how to describe this interaction between particles' spins. From this perspective, the spins are arranged in a graph, usually a lattice. The spin's influence is measured by the distance over which the direction of one spin affects the direction of its neighbor's spins. This distance is called the correlation length; it has an important function in the identification of critical phenomena. Indeed, the correlation length measures the distance over which the behavior of one microscopic variable is influenced by the behavior of another. Away from the critical point (at low temperatures), the spins point in the same direction. In such a situation, the thermal energy is too low to play a role; the direction of each spin depends only on its immediate neighbors making the correlation length finite. But at the critical point, when the temperature has been increased to reach the critical temperature, the situation is completely different. The spins no longer point in the same direction because the thermal energy dominates the whole system and the magnetization spin-spin vanishes. In this critical situation, spins point in no specific direction and follow a stochastic distribution. Moreover, each spin is now influenced by all other spins (not only its neighbors) whatever their distance. This situation is a particular configuration in which the correlation length is very important (it is considered to be infinite). At this critical state, the whole system appears to be in a homogeneous configuration characterized by this infinite correlation length making the system scale invariant. Consequently, the spin system has the same physical properties whatever the scale length considered. It is worth mentioning that large variation correlation length appears to be ruled by a power-law, which is also a key element of most econophysic models.[19]

The Ising model is a perfect illustration of the autonomous nature of mathematical model used as an empty container that characterizes the mathematical analogies. Interestingly, this model is used by Knuuttila and Loettgers (2016) as a telling example of a model template. As they showed, the transfers between the disciplines allowed by the Ising model are "not just based on its computational aspects" (Knuuttila and Loettgers 2016, p. 397). Precisely, at the origin, the Ising model is a mathematical model of ferromagnetism used to study phase transitions and critical points. However, "it 'possesses no ferromagnetic properties'" (Hughes 1999, p. 104). Its abstract and general structure permits its use to the study many other problems or phenomena characterized by phase transitions and critical points:

> The Ising model is employed in a variety of ways in the study of critical point phenomena. To recapitulate, Ising proposed it (…) as a model of ferromagnetism; subsequently it has been used to model, for example, liquid-vapour transitions and the behaviour of binary alloys. Each of these interpretations of the model is in terms of a specific example of critical point behavior (…). [T]he model also casts light on critical point behaviour in general. Likewise, the pictures generated by computer simulation of the model's behaviour illustrate (…) the whole field of scale-invariant properties (Hughes 1999, pp. 124-25).

[19] A power-law distribution is a special kind of probability distribution, such as $p(x) = Cx^{-\alpha}$. Such distribution is leptokurtic, and it is widely used for studying variables that have extreme values. In economics, a well-known power law is the Pareto law. The main property of power laws is their scale invariance. Moreover, from the viewpoint of statistical physicists, power laws are synonymous with complex systems, which makes their studies particularly valuable.

For these reasons, the Ising model is one of the most simplified descriptions of a system with a critical point[20]; it has played a central role in the development of research on critical phenomena (Jovanovic and Schinckus 2017; Sornette 2014). "The specification of the [Ising] model has no specific physical content" (Hughes 1999, p. 99); its content is mathematical. Therefore, as section 2 explained, Ising model is independent of the underlying phenomenon studied, and it can be used to analyze any phenomena that share the same mathematical characteristics. In other words, the Ising model is a telling example of a mathematical model used in the process of mathematical analogies, which has allowed the transfers from physics to economics.

How did statistical physicists proceed? As Jovanovic and Schinckus (2017, p. chap. 3) explained, statistical physicists adopt a phenomenological methodology. They identified phenomena with large numbers of interacting units whose microscopic behaviors could not be observed directly but which can generate observable macroscopic results. Therefore, modelers can look for statistical regularities often characterized by power laws, which are the main mathematical characteristic of critical phenomena, and also the main elements for justifying the application of statistical physics models, like the Ising model, to new phenomena and new target domains. As section 2 explained regarding the process of mathematical analogies, we have one mathematical model, which is the source domain and an empty container, and with a multiplicity of target domains. Moreover, it is worth saying that the process of identification used during this application of mathematical analogies is similar to Ménard's second step: the target domain is "modeled" by the analogical reasoning. For instance, any work in econophysics advances empirical results to demonstrate that the phenomena studied are ruled by a distribution of variables or observables following a power law scaling, then varies (*i.e.* calibrate) the parameter α of the power laws until they obtain the best fit between the graph of the power law and the one describing the empirical data.[21] However, power laws can visually be close to so-called exponential laws, and it is extremely difficult to distinguish between them.[22] Therefore, this process of identifying a power law in the target domain is a telling illustration of how the domain source creates similarities in the target domain. At this stage, in the early works in econophysics, the mathematical analogies are similar to what we observed with Frisch: we have one source domain (*i.e.* physics) for the mathematical model.

To conclude this section, Frisch's model and the Ising model illustrate the diffusion of the use of mathematical analogies for studying economic phenomena. In accordance with Israel's analysis, these analogies drop the restriction that the source domain has to be mechanics or physical mechanics and its underlying causalities. However, the mathematical model used in the analogical process still has a physical meaning. Indeed, in the case of the oscillator, the formalism is mathematical, but there is a physical basis. It is similar to the Ising model. In other words, physics remains a justification for the use of mathematical analogies. However, as the next section details, the econophysicists' approach uses the mathematical analogies differently.

4. MATHEMATICAL ANALOGIES: ENGINES FOR MAKING THE DISCIPLINARY BOUNDARIES OF ECONOMICS AND PHYSICS MORE PERMEABLE

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[20] As Knuuttila and Loettgers (2016) remind, "The Ising model is one of the most simplified and successful models in physics" and it is used as an example of a minimal model by Weisberg (2007).

[21] In the same vein, econophysicists introduced truncation techniques for power laws in order to fit the observations of the target domain with their mathematical model (Jovanovic and Schinckus 2017, p. chap. 3).

[22] In fact, many of the power laws that econophysicists been trying to explain are not power laws at all (Clauset *et al.* 2009).

The last step of our analysis exposes recent interactions and avenues that emerge with econophysics. Jovanovic, Mantegna, and Schinckus (2019) showed that some models developed by econophysics are today being brought back into physics. They analyzed three illustrations of such transfers from economics to physics: the modeling of out of equilibrium processes, signal detection in multivariate systems and information process and aggregation in multi-agent physical systems. This section analyzes the way mathematical analogies have created opportunities to use an economic model for solving physical issues and show some evolutions in mathematical analogies. For this purpose, we use the so-called "minority game".

4.1. Minority Game: An Illustration of Transfers from Economics to Physics

Minority game was created mainly by two econophysicists, Challet and Zhang (1997). It is a stylized version of the "El Farol bar" problem. This is a well-known problem in game theory, originally introduced by an economist, W. Brian Arthur in 1994.[23] At the 1994 AEA annual meeting, this author presented the following problem. *N* agents decide independently each week whether to go to a bar, named El Farol, that offers entertainment on a certain night. Each agent goes if he expects fewer than 60 people to show up or stays home if he expects more than 60 to go. The choices are unaffected by the previous visits; there is no collusion or prior communication among agents; the only information available is the numbers who came in the past weeks. No agent knows the model on which other agents base their predictions. Arthur (1994) concluded that in this context there is no deductively rational solution based on the theoretical hypotheses of economics: if all believe few will go, all will go, invalidating this would invalidate that belief; similarly, if all believe most will go, nobody will go, invalidating that belief. In other terms, the "El Farol bar" problem provides an illustrative example of the process of rational decision between two alternatives of a group of rational agents with the presence of negative externalities. In this setting, there is no self-fulfilling equilibrium and therefore in assuming fully rational use of the public information the system oscillates between states that are always frustrating for the agents.

Despite this evidence, Arthur decided to use computer experiments for searching if an equilibrium emerges. In his experiments, 100 agents use randomly use on strategy among k strategies. If the strategy does not work, the agent drops it for the future, otherwise he keeps it. His computer experiments showed a possible Nash equilibrium: on average 40 percent forecast above 60, 60 percent below 60. In other words, Arthur was able to show that a suboptimal (economic) equilibrium occurring at each time step around an a priori optimal allocation of the resource is reached by the system by hypothesizing a bounded rational inductive reasoning of the agents, but was not able to obtain a solution for such a situation "that is both evolutionary and complex" (1994, p. 411).

After demonstrating his economic equilibrium problem to economists, including economists specialized in game theory, Arthur had to admit that they did not know how to deal with it. He also noted that econophysicists found:

> economists didn't quite know what to make of [my paper]. My colleague at Santa Fe, Per Bak, […] saw the manuscript and began to fax it to his physics friends. The physics community took it up, and in the hands of Challet, Marsili and Zhang, it inspired something different than I expected –the Minority Game. El Farol emphasized [for me] the difficulties

[23] Game theory was founded by the mathematicians von Neumann thanks to the publication of his *On the Theory of Games of Strategy* in 1928, but it was his book published with the economist Oskar Morgenstern that initiated research in this field (Leonard 1995).

> of formulating economic behavior in ill-defined problems. The Minority Game emphasizes something different: the efficiency of the solution (Arthur 2005, p. Vii).

In fact, Challet and Zhang (1997) formalized the "El Farol bar" by using a mathematical model developed by Caldarelli, Marsili and Zhang (1997) for studying the dynamics of financial market prices. This econophysic financial model draws also draws inspiration directly from the challenge Arthur addressed with the El Farol bar problem. Caldarelli et al. (1997, p. 479) observed that "from a physicist's point of view, the market is an excellent example of self-organized systems: each agent decides according to his own perception of the events" and "the system reaches dynamically an equilibrium state characterized by fluctuations of any size, without the need of any parameter fine tuning or external driving."[24] In other words, they were able to obtain a self-fulfilling equilibrium for the El Farol bar problem. To obtain such an equilibrium, they introduced new hypotheses. Firstly, their reasoning is based on the typical financial economics equilibrium, the absence of arbitrage opportunity, which is not the typical economic market equilibrium. In this case, without new information, the price fluctuation dynamic should be stationary. Secondly, they analyze the El Farol Bar problem as an application of a Darwinist selection: at each time step, the agent with the smallest capital is eliminated and replaced by one with a new (random) strategy. From this perspective, the agents trade in a non-ending fight against each other in order to survive. Thirdly, they introduced the hypothesis of a power law to describe the distribution of the wealth of the traders. The numerical simulations of their model show close resemblance to the fluctuations of the Standard & Poor's 500 index or to high frequency foreign exchange data. This result suggests that the stock market equilibrium is mainly due to the interaction among "speculators," regardless of economic fundamentals. Moreover, and in accordance with Arthur's problem, their model does not contain adaptive dynamics in the players' strategies.

Challet and Zhang (1997) kept the hypothesis of a Darwinist selection for solving the El Farol problem: "the poor players are regularly weeded out from the game and new players are introduced to replace the eliminated one" (1997, p. 415). In other words, some agents are discouraged and give up going to the El Farol bar in the future. To keep a certain diversity, they also introduce a mutation possibility in cloning. They also allow one of the strategies of the best player to be replaced by a new one. With these additional hypotheses, they observe that this population is capable of "learning." The learning process is demonstrated by their simulations. Specifically, they show that if we increase the information available used by the agents (*i.e.* number of past results about the numbers who came in the past weeks), the volatility decreases, meaning that the agents make the right choice more often. Moreover, agents who use less information underperform compared to the agent who use more information. "Remarkable is that each player is by definition selfish, not considerate to fellow players, yet somehow they manage to better somewhat share the limited available resources" (Challet and Zhang 1997, p. 409). "The players manage to defy entropy, in other words to get themselves organised to occupy less unlikely configurations" (1997, p. 412).

The Challet and Zhang (1997) and Caldarelli, Marsili and Zhang (1997) models use many key concepts of statistical physics, like entropy, power laws, "self-organized criticality," etc. The minority game is therefore a telling example how econophysics provides a solution to a problem that was originally conceived of in economics and was not tractable in terms of classic game theory with classic economic hypotheses.

[24] "Self-organized criticality" is another key econophysics' concept (Jovanovic and Schinckus 2017, p. chap. 5) Self-organized criticality is the property of dynamic systems that naturally self-organize into a critical point while obeying a power-law.

The minority game has another interesting feature. While it was created in order to analyze an economic problem, this model has then been applied back to physics and some related fields. It has been used for instance in radio engineering and computer science in order to improve wireless networks (Mähönen and Petrova 2008) or to improve coordination in wireless sensor networks (Galstyan *et al.* 2004). In computer science, the minority game is used to improve the heterogeneous Delay Tolerant Networks (Sidi *et al.* 2013). According to Mähönen and Petrova (2008, p. 100), it could also be applied for studying the behaviors of flocks of birds. In other words, the minority game is an empty container that is applied to various phenomena totally independent of the economic phenomena in which it originated. It is worth noting that, in contrast to Van der Pol or Le Corbeiller, Chalet and Zhang did not feel it necessary to give a list of possible applications. To us, this is due to the fact that creating mathematical models using a mathematical analogy is today a common practice.[25]

4.2. Minority game and mathematical analogies

Our investigation shows that the minority game illustrates a noticeable change in the mathematical analogies. This mathematical model has been developed by borrowing simultaneously from economics and physics.[26] We noted that it integrates many key concepts of physics. However, it also integrates many key concepts of economics, such as the absence of arbitrage opportunity, rationality, cooperative games, and externalities. In other words, this mathematical model combines concepts and formalisms from economics and physics, which are both its source domains. In contrast, mathematical models saw in section 3 have only one source domain. However, they still have multiple target domains.

In our opinion, we can also find some explanations in the institutional position of econophysics which provides important clues for this transformation. Econophysics has a singular institutional position: outside economics and in the shadow of physics (Gingras and Schinckus 2012; Jovanovic and Schinckus 2017, p. chap. 4). This position reflects the autonomy of the field and has created the opportunity for developing new hypotheses and models outside physics and economics (Jovanovic and Schinckus 2017). But it is also compatible with the use of the mathematical analogies we have outlined. In the case of minority games, the mathematical models which have been developed by econophysicists become more autonomous, and have permitted transfers to both economics and physics. In other terms, there is not one source domain anymore, but two source domains (*i.e.* physics and economics) for the mathematical model and several target domains. This result is compatible with some changes in econophysicists' perspective. Today they defend a "mutual fertilization"[27] between both disciplines rather than a unidirectional influence of physics on economics, as it was common in the past. This position is compatible with the evolution of the discipline, as Jovanovic and Schinckus (2017) have pointed out: it is becoming increasingly a transdisciplinary one. To conclude, these new perspectives are in line with the change we pointed out: from mechanical analogies –restricted to one source domain that concerns mechanical phenomena– to mathematical analogies that make room for the kinds of two-way inferences between economics and physics (*i.e.* as an isomorphism), because the emphasis is on the reproduction of phenomena rather than causal explanation as Israel suggested.

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[25] This observation is compatible with Claveau (2019): articles published in economics journals cited fewer and fewer articles published in mathematics journals, suggesting that mathematical models used by economists are more and more created within economics.

[26] We detailed the example of the minority game. However, it is not the only one. For instance, physicists nowadays are using GARCH models (Modarres and Ouarda 2014).

[27] This expression is borrowed from Sornette (2014, p. 1).

5. CONCLUDING REMARKS

Our investigation has shown how "mathematical analogies" have contributed to making the disciplinary boundaries between economics and physics more permeable. These boundaries were drawn before the twentieth century, while mathematical analogies emerged during the 20th century. As we have shown, the transformation of economics' boundaries since the 1930s is directly concerned with the main characteristics of mathematical analogies. The latter are based on an autonomous mathematical structure that is fueled by concepts and formalisms that come from both disciplines. Our investigation does not pretend to be exhaustive, but it offers a perspective on the relations between economics and physics. Further investigation will be required to ascertain if this emergence process is in fact transforming the boundaries between these disciplines.

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