A unified model for price return distributions used in econophysics

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I. Introduction

For the 1970s, a new theoretical movement has been initiated by some physicists who began publishing articles devoted to the study of social phenomena, such as the formation of social groups (Weidlich [1]) or social mimetism (Callen and Shapiro [2]). The next decade confirmed this new theoretical trend (labelled *sociophysics*), as the number of physicists publishing papers devoted to the explanation of social phenomena and the number of themes analyzed continued to increase. During the 1990s, physiciststurned their attention to economics, and particularly financial economics, giving rise to econophysics. Although the movement's official birth announcement came in a 1996 article by Stanley *et al*. [3], econophysics was at that time still a young and ill-defined field. Econophysics can be defined as "a quantitative approach using ideas, models, conceptual and computational methods of statistical physics". Today, econophysics is becoming an institutionalized field (Gingras and Schinckus, [4]) with different journals proposing a prolific literature about the way of characterizing the evolution of financial prices.

There is an "extreme diversity" of models recently developed by econophysicists (Rickles [5]) and many theoretical frameworks still emerge. This diversity refers, among other things, to the presence of a large number of different models for the returns distribution function, which is due partly to the market situations that are related to human behaviour and therefore, are quite difficult to describe.

Indeed in reference [6] it is shown that the shape of the return distribution is not unique and depends upon the state of the market; more precisely it depends upon the interaction between agents doing transactions. Since the intensity of these interactions is different from market to market, so is the return distribution. Another cause of diversity may be the relative youth of the Econophysics, which makes that in some situations there is no total agreement between different authors.

In this paper, our objective is to provide a global parameterised framework that includes all econophysics models and would apply in any market situation by a change of parameters . Indeed, the standardization of knowledge through a common scientific culture is a necessary condition to become a strong discipline (Kuhn [7]). We propose a generic formula characterizing the statistical distributions usually used by econophysicists (Levy, Truncated Levy or no stable Levy distributions). Such formula will contribute to unify econophysics and to base this new field on shared scientific standards since the possibility to find a generalized formula is derived from the common conceptual tools shared by econophysicists. This will enable econophysics be no longer an ill-defined field. Moreover, such generalized formula allows a systematic comparison between the different models used by econophysicists.

II. **Generalized formula for price return distributions.**

For describing the probability distributions of stock market price changes, many models using different types of probability functions are proposed in the econophysics literature. However, Gringras and Schinckus [4] showed that *Physica A* appears to be the leading journal and that Mantegna, Bouchaud, Mandelbrot, Sornette and Lux are the most cited authors in econophysics. Our analysis is based on these results. We also add other important authors such as Stanley, Gopikrishnan or Plerou who are also very cited authors in econophysics (*Web of Science*) and others cited in the references. Among the authors identified, we have selected econophysics papers dedicated to modelling the distribution of price returns (generally econophysicists make use of logarithmic returns).

From the study of the references indicated at the end we have reached to the following generalized density distribution formula:

$$P(x) = Cf(x)e^{-g(x)+d} \qquad (1)$$

where x is the log-return, C and d are constants that might have temporal variation.

The analytical form of $f(x)$ is not always known for all the values of x[9], but it has a power law variation, at least in the limit of large x. For low x values $f(x)$ refers most often to the Lévy stable distribution.

$$f(x) \sim \frac{1}{x^{b_1 + a_1 \alpha}} \ \ for \ |x| \to \infty \tag{2}$$

a_1 and b_1 are two parameters (usually equal to 1) that define the shape of the distribution at large x ,and α is the principal exponent of the power law. The function g introduced in equation (1) has the form:

$$g(x) = (a_2 x + b_2)^{c_2} \tag{3}$$

In equation (3) a_2, b_2 and c_2 are parameters that are different from one model to another, defining the final shape of the distribution function. Finally the generalized density distribution is:

$$P(x) \sim \frac{e^{-(a_2 x + b_2)^{c_2} + d}}{x^{a_1 \alpha + b_1}} \tag{4}$$

A first commentary that we can do concerning this formula is the big number of parameters (7) that had to be introduced in order to be consistent with our purpose to include all important studies. This excessive parameterisation is due to the great diversification of models presented in literature. But as we'll see further, in the great majority of the models a_1, b_1 are 1; b_2 and d are equal to zero.

A common characteristic of the distribution functions used by econophysicists refers to a power law variation for large values of x (Pareto law). However, the exponent of the power law differs from one authors to another [9].

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[9] Most often f(x) refers to the Lévy stable distribution whose analytical shape is known only in some specific cases.

III. **Application:**

This formula allows to rewrite and to compare the distribution of price changes used in the main econophysics models. This section, for providing a classification of the main econophysic models uses this formula. Three classes of econophysics models are considered depending the distribution used (Levy stable, Truncated Levy or non stable Levy distribution).

III.1. **Lévy stable distributions**.

The simplest case of Lévy stable distribution is the Gaussian distribution; it is also a simple particular case of equation (5) and can be obtained by imposing $C = \frac{1}{\sqrt{2\pi\sigma^2}}$, $a_1 = b_1 = 0$, $b_2 = d = 0$, $c_2 = 2$ and $a_2 = \frac{1}{\sqrt{2\sigma^2}}$.

$$P(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/(2\sigma^2)} \quad . \tag{5}$$

Gaussian distribution is intensively used in neoclassical finance for describing price variations. With x being the log-returns the equation (5) defines the lognormal distribution.

The use of lognormal law in finance was introduced by Osbone [9] in order to avoid the theoretical possibility to have negative prices. Moreover, this use is also based on the assumption that the rate of returns rather than the change of prices, are independent random variables.

Unfortunately this distribution does not describe correctly the empirical data especially for high price variations. With a Gaussian distribution, the probability of having very high price variations is much lower than what is observed in real data, thus the appearance of financial crashes is highly underestimated (McCauley [10]).

Because econophysicists adopt an empiricist perspective (Schinckus, [11]), they are looking for the distribution functions *P(x)* that must fit empirical financial data without a priorism. Sometimes, distribution functions are directly derived from physics models used for describing stochastic dynamic process.

The general Lévy stable distributions functions were first proposed by Mandelbrot [12] and used afterwards by the pioneers of econophysics since they describe better the tail of the distribution of financial data then a Gaussian distribution. Most important, for large x, the Lévy stable distributions are well approximated by a power law as described in equation (2) with the exponent α having values between 1 and 2, generally around 1.5. Besides pure Léby stable models we have put in this class some models distributions that approach a Lévy stable one only in the limit of high returns values.

When compared to equation (5) most of the authors propose distributions with $a_1 = b_1 = 1$ [6, 13-18] with one special case where $a_1 = -1$ and $b_1 = 1$ [33]. The parameter a_2 is non zero only in three cases [`3, 13, 18] and $c_2 = -1$ in [13]. The others parameters of equation (5) are always taken to be zero in this case. The observed non-zero a_2 involves the presence of an exponential term in the distribution function P(x) which is derived by using models to explain the empirical data (for example, generalized Lotka Voltera model [13] and the Percolation model [6, 18]). For example the model of distribution proposed in reference [6] is a pure power law in the presence of large interacting clusters, but it is an exponentially moderated power low in a market with non-interacting agents. However,most often the authors focus to calculate the power law exponent of the distribution tail in some specific situations.

The main drawback of Lévy stable distributions is that they have infinite variance, a situation that in physics cannot be accepted. As Gupta and Campanha [18] point out, "Lévy flight have mathematical properties that discourage a physical approach because they have infinitie variance". Physicists have chosen to characterize financial phenomena through Lévy processes but they explicitly reject the idea of infinite variance. In this perspective, some physicists have developed statistical methods in order to truncate the Levy stable distribution.

III.2 **Truncated Lévy distributions**

It is always preferable to use distributions with finite variance for describing the stock price variations. Two reasons can be evoked for this: on one hand, the fact that a finite variance is more in line with a physical approach and, on the other hand, this notion of variance usually refers to the idea of risk in finance.

In order to solve the problem of infinite variance econophysicists developed truncated Lévy distributions. These distributions are Lévy stablethat have a cut-off length for the price variations above which the distribution function is set to zero in the simplest case [19], or decreases exponentially [18, 20,21]. These functions are chosen in order to obtain the best fit with the empirical data.

We can find these truncated distributions in our generalized formula when $a_1 = b_1 = 1$ and at least a_2 and c_2 different of zero (b_2 is nonzero in ref [18, 21].

The ref. [22] gives a distribution function with the *d* constant non-zero, thus it is a power law with exponential decrease on the whole range of returns values, that does not has any specific cut-off length. For the simply truncated distribution from ref. [19] one can consider a_2 very large (going to ∞) beyond the cut-off length.

III.3 **Non-stable Lévy distributions.**

Some empirical studies about financial markets suggested that Lévy stable could overestimate the presence of large price variations even though they are much closer of data than a Gaussian (Gabaix and al. [9]).

In order to solve this point, some authors have proposed a power law variation of $P(x)$ for large x but with the values of the exponent α greater than 2. The parameters of equation (4) are in this case $a_1 = b_1 = 1$ [23-25] with one special case where $a_1 = -1$ and $b_1 = 1$ [25]; $a_2 = b_2 = c_2 = 0$ except for [26] where a_2 is non-zero and $c_2 = 1$. In this last case the authors have used the Focker Planck equation for anomalous diffusion to derive the probability function. A Lévy non-stable distribution for describing price variations is also obtained as a special case of a Tsallis distribution derived in ref. [27]. One should also note that for large *x* a student distribution used in ref. [28] approaches Lévy distributions.

Finally we specify that there are models that cannot be includes in these three classes. In ref. [29-31] authors proposed exponential distribution functions in terms of logarithmic returns for the intraday trading of bonds and foreign exchange. It is found, however, that these models work well only in a restricted region of return values. They are also special cases of the probability density given by equation (5) with $a_1 = b_1 = d = 0, c_2 = 1$.

Thus we see that there are three main categories of distribution proposed in the econophysics literature (with few exceptions that have been specified), having as a common point a power law variation whose exponent varies from one category to another: the value of α is between 1 and 2 for the stable and truncated distributions and above 2 for the unstable distribution. There are proposed models that are not pure Lévy stable derived power laws, but only approach to such laws in the limit of high returns values. The truncated Lévy distributions are Lévy stable distributions that above a cut-off length are truncated to zero or decrease exponentially.

IV. **Conclusion and implications**

We acknowledge that no simple function can perfectly uniquely describe the financial data. The generalized distribution given by equation (4) is a "meta-equation" derived from the main models used in econophysics and which describes well the empirical data at larges values of *x* (with the mention that there are few non-determined parameters). Econophysicists want to describe the financial phenomena as they are and not as they should be [32]. In this empiricist perspective, they want to go beyond the Gaussian framework because financial data cannot be empirically described by a Gaussian distribution. In order to describe the complexity of financial data in a more realistic way, econophysicists had then to develop more sophisticated tools. Therefore, they developed or proposed different Levy processes for which this paper provides an unified framework.

Our generalized formula contributes to structure econophysics such as a scientific discipline with a clear method and a common scientific culture. This conclusion is directly in line with the biblometric and sociological conclusions given by Gingras and Schinckus [4] concerning the strong institutionalization of econophysics. In this perspective, econophysics appears more and more as specific field independent from economics with a “lack of awareness of work that has been done within economics” (Gallegati and al. [33,p.1]). Our formula also helps to overcome some limitations of econophysics to become the next dominant paradigm in financial theory, such as identified in the analysis related to econophysics done by Jovanovic and Schinckus [34].

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